7

QUEEN'S UNIVERSITY SERIES.

INTRODUCTION TO THE SCIENCE

OF

DYNAMICS.

BY

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KINGSTON, ONT.:

PRINTED BY WILLIAM BAILIE.

1886.

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INTRODUCTION.

Mathematics may be defined as the science of measurement. It is divided generally into (1) Pure Mathematics, and (2) Applied Mathematics. In the former, measurements of *space* and *time* alone are considered. In the latter, besides space and time, the properties and conditions of *matter*, such as mass, weight, energy, temperature, potential, are measured. In a wider sense Applied Mathematics is known as *Natural Philosophy* or *Physics*. Natural Philosophy is the science which investigates and measures the properties of matter as discovered by direct observation and experiment and deduces the laws connecting these properties. So extensive, however, has our knowledge of the properties and conditions of matter become, that different branches of Natural Philosophy are conveniently separated from the parent stem. Chemistry, Astronomy, Geology, Physiology, &c., though originally branches of Natural Philosophy, have put forth roots like the branches of the banyan tree and become themselves trees of knowledge, sending forth their own branches, and these in their turn new roots. But the same *vital force* permeates trunk and branches alike, and it is this vital force, under its new name *energy*, which now forms the subject-matter of physical science. Natural Philosophy or Physics is thus the science of energy and is divided into the following principal divisions: (1) Dynamics, which treats of plainly visible energy, (2) Sound, (3) Heat, (4) Light, (5) Electricity and Magnetism.

Before any measurements can be made, certain units of measurement must be fixed upon. Thus, the navigator measures the run of his ship in knots, the surveyor his land in acres, and states of heat are measured in thermometric degrees. Now, not only in different countries, but even in the same country, different units, bearing no simple relations to one another, are constantly used in measurements of the same kind. In order to avoid all unnecessary calculations in the comparison of different observations, scientific men have agreed to adopt a uniform system of units. This is founded on the French system of units and is known as the *Centimetre-Gram-Second* or C. G. S. system. In the following pages the student is therefore exercised in the use of the C. G. S. as well as the English units.

When for special measurements it is desirable to use larger or smaller units than the standards, these are formed in the C. G. S. system quite uniformly, except in measurements of *time*, by prefixing the words *deca*, *hecto*, *kilo*, *mega*, to the name of the standard to indicate multiples of 10, 10^2, 10^3, 10^6, times the standard unit, and by prefixing *deci*, *centi*, *milli*, to indicate submultiples of 10^{-1}, 10^{-2}, 10^{-3}. Taken in connection with the decimal notation in the writing of numbers, such a system of forming the multiples and submultiples saves all unnecessary calculations in reducing to the standard unit. In the English system of forming multiples and submultiples, the numbers seem to have been chosen with a view of containing as many prime factors as possible, an imaginary advantage which has occasioned a very great amount of unnecessary calculation. In conformity with the scientific system of units, all temperatures in the following pages are given in degrees *centigrade*.

Units of Length.

10^3 millimetres = 10^2 centimetres = 10 decimetres = 1 metre = 10^{-1} decametre = 10^{-2} hectometre = 10^{-3} kilometre = 10^{-6} megametre.

3 feet = 1 yard, 6 feet = 1 fathom, 100 links = 1 chain = 22 yards, 5280 feet = 1760 yards = 80 chains = 1 mile.

Units of Surface.

1 are = 1 square decametre = 10^2 square metres = 10^6 square centimetres.

1 acre = 10 square chains, 640 acres = 1 square mile.

Units of Volume.

1 litre = 1 cubic decimetre = 10^3 cubic centimetres.

1 gallon = 277 cubic inches nearly, and holds 10 lbs. avoir. of water at 62° F.

Units of Mass.

1000 milligrams = 100 centigrams = 10 decigrams = 1 gram = 10^{-1} decagram = 10^{-2} hectogram = 10^{-3} kilogram = 10^{-6} tonne.

1 pound avoirdupois = 7000 grains, 1 ton = 2240 pounds.

The numerical relations between the C. G. S. and English units are given on pp. 107–110.

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INTRODUCTION TO THE SCIENCE OF DYNAMICS.

CHAPTER I.

Matter. *Extension.*

1. The student of elementary dynamics is not concerned with the ultimate structure of matter, of which various theories have been advanced by scientific men, but only with its properties. The principal of these which we shall consider are extension, inertia, mass, weight, and energy.

2. Any portion of matter is called a *body*. The grains of sand on the sea-shore, our own bodies, houses, the whole earth, the planets, the fixed stars, &c., are examples of bodies. The expression of the fact that two or more bodies cannot at the same time occupy the same portion of *space* is known as the principle of *impenetrability*.

3. *Extension* is that property of matter in virtue of which every body occupies a limited portion of space. This implies that every body has *form* or *shape*. The *volume* of a body is the measure of its extension. The term *bulk* is often used in the same sense. The internal volume of a body, *e.g.* that of a cup or of a hollow sphere, is the amount of space enclosed by the body, and is called its *capacity*.

4. Before any measurements can be made it is necessary to fix upon *units* or definite quantities of what we desire

to measure. In terms of these units we express by numbers any other quantities of the same kind. In measurements of extension four units are used, viz., units of length, area, volume, and angle. Of these one may be taken as a *fundamental unit*, and the others made to depend upon it, and these are hence called *derived units*. It is most convenient to take the unit of length as the fundamental unit.

5. The English unit of length (or distance) is the yard, and is defined by Act of Parliament as the distance between two points on a bar of metal at a definite temperature. The French unit, the *metre*, although derived originally from the supposed dimensions of the earth, is similarly defined. The unit of length adopted by scientific men is one of the submultiples of the French unit, viz., the *centimetre*, and its multiples and submultiples are the same as the French, viz., millimetre as submultiple, and decimetre, metre, decametere, hectometre, kilometre as multiples.

6. Whatever unit of length be used, it is found most convenient in measurements of surface to take as the unit of area (or surface) the area of a square of which the side is unit of length, or a multiple or submultiple thereof. Hence the scientific unit of area is a *square centimetre*. The French unit of area, the *are*, is a square decametre, and therefore equal to 10^6 scientific units of surface.

7. Similarly the unit of volume is immediately and most conveniently derived from the unit of length by defining it as the volume of a cube of which the edge is unit of length or a multiple or submultiple of the unit of length. Hence the scientific unit of volume is a *cubic centimetre*. The French unit of capacity, the *litre*, is a cubic decimetre, and therefore equal to 10^3 scientific units of volume.

8. The unit of angle in common measurements is the *degree*, which is the 90th part of a right angle. It may seem strange to say that the unit of angle can be derived from the unit of length, but this will be understood when we remember that if a circle be described with the vertex of the angle as centre and with any radius, the magnitude of the angle is measured by the ratio of the length of the arc on which it stands to the length of the radius. We may, therefore, define the unit angle as that angle which is subtended by an arc of unit length at the centre of a circle of unit radius. This is just the same as an angle which is subtended by an arc, whose length is equal to the radius, at the centre of any circle whatsoever. Such an angle is the scientific unit of angle, and is called a *radian*.

9. As mentioned above, we learn from elementary geometry that, whatever units of length and angle be adopted, the following formula expresses the relation between the length of any arc (a) of a circle, the length of the radius (r), and the magnitude of the angle (θ) subtended at the centre of the circle by the arc

$$a = C r\theta$$

where C is a constant number, whose value depends upon the unit of angle adopted. If we measure θ in radians, this reduces to the simple form

$$a = r\theta.$$

10. *To express the value of a radian in degrees.* Since the arc subtended by two right angles $= \pi r$, if θ be the measure of two right angles in radians, we get $\pi r = r\theta$, $\therefore \theta = \pi$, *i.e.* two right angles $= \pi$ radians, and $\therefore$ a radian $= \dfrac{180^\circ}{\pi} = 57^\circ\ 17'\ 44''\cdot 8$ true to the tenth part of a second of angle.

11. In many dynamical investigations it is unnecessary to consider in any way the dimensions of a body, or the distances between the different parts of a system of bodies. When this is the case, the body or system of bodies is called a *particle* or *material particle.* Thus in the explanation of the seasons, or of the phases of the moon, the earth or moon is a body, as we cannot neglect its dimensions, whereas in the determination of the sun's position in the ecliptic at any time it is a *particle.* In considering the proper motion of the solar system amongst the fixed stars, the sun, and indeed the whole solar system, are merely particles. The term particle is frequently defined as an indefinitely small body. The terms small and large are merely relative, and what is small at one time or from one point of view is large at another or from another point of view.

12. The most generally accepted theory of the ultimate structure of matter at the present day is known as the atomic theory. According to this theory matter is not infinitely divisible, but consists ultimately of excessively small indivisible particles. The smallest portion of any body, beyond which mechanical sub-division is supposed to be impossible is called a *molecule.* A molecule may, however, be *chemically* divided into *atoms.* Thus a molecule of water (H_2O) may be chemically divided into three atoms, two of Hydrogen, and one of Oxygen.

Examination I.

1. Give various examples of *matter.*

2. Distinguish between the terms *body*, *particle*, *molecule*, *atom.*

3. Define *extension*.

4. Distinguish between *volume*, *bulk*, and *capacity*.

5. What is a unit of measurement? Give examples.

6. What is a metre, a litre, a radian?

7. Express in radians the angles of an equilateral triangle, half a right angle, $30°$, $75°$, 4 right angles.

8. If the unit of length be a foot, and the unit of angle a right angle, what must be the value of C in the formula $a = Cr\theta$?

9. Is the value of C in the above formula dependent on the unit of length? Why?

Exercise I.

Note.—The following examples in mensuration are appended to exercise the student in the use of the new units and also of logarithmic tables to which he should early accustom himself.

1. The great pyramid of Gizeh is a regular pyramid on a square base. The original length of an edge of the base was 22042, and of a slant edge 23286·5; find (1) the area of the ground on which it stands, (2) the exposed area of the pyramid, (3) the volume.

2. Assuming the earth to be a sphere, and that the length of an arc of a degree on a meridian is equal to $1{\cdot}1119 \times 10^7$, find (1) the length of the diameter, (2) the area of the earth's surface, (3) the volume.

3. If the nature of the earth's crust be known to a depth of 8 kilometres, find the total volume known, and the ratio of the known to the unknown volume, supposing the earth to be a sphere of 6370·9 kilometres radius.

4. On the same supposition, how much of the earth's surface could a person see who was at a height of 4 kilometres above the sea level?

5. If the atmosphere extend to a height of 70 kilometres, what is the ratio of its volume to that of the solid and liquid earth?

6. Compare the surfaces of the torrid, temperate, and frigid zones of the earth, supposing the first to extend to an angular distance of 23° 30′ from the equator, and the last to a distance of 23° 30′ from each pole. Determine also in square kilometres the amount of surface in each.

7. The sides of a triangle are 3 metres 3 centimetres, 2 metres 8 decimetres, and 2 metres 1 decimetre 4 centimetres; to determine its area and greatest angle.

8. Find the slant edge, surface, and volume of a circular cone, the diameter of its base, and its height being each 1 metre.

9. Two sectors of circles have equal areas, and the radii are as 1 to 2; find the ratio of the angles.

10. A gravel walk of uniform breadth is made round a rectangular grass-plot, the sides of which are 20 and 30 metres; find the breadth of the walk, if its area be three-tenths of that of the grass-plot.

11. The diagonals of a parallelogram are 8 and 10 metres, and its area one-third of an are; find the angle between the diagonals.

12. Find the number of litres of air in a room whose dimensions are $12\frac{1}{2}$ m., 5·45 m., and 3·7 m.

13. Find the angle of a sector of a circle, the radius of which is 20 metres, and the area a deciare.

14. The diagonals of a parallelogram are to one another as sin $\frac{\pi}{3}$ to sin $\frac{\pi}{6}$, prove that the figure is a rhombus.

15. The sides of a quadrilateral, taken in order, are 7·5, 5·5, 6, and 4 metres, and the angle between the first two is $74^\circ\ 40'\ 15''$; shew that the figure may be inscribed in a circle and find its area.

16. The horizontal parallax of the sun (*i.e.* the angle subtended by the earth's radius at the sun) is $8''\cdot 85$, and of the moon $57'\ 3''$; find the distances of these bodies in terms of the earth's radius.

17. Find also, in terms of a great circle of the earth, the areas of the moon's orbit and of the ecliptic, supposing these to be circles.

18. Find the circumference and area of the circle of latitude passing through Kingston, Ont., latitude $44^\circ\ 13'$.

19. A pendulum whose length is $1\frac{1}{2}$ metre swings through an arc whose chord is a decimetre; find the angle and the length of the arc of oscillation.

20. Prove that the area of a regular polygon of n sides, inscribed in a circle of radius r, is equal to $n\frac{r^2}{2}\sin\frac{2\pi}{n}$; hence find the areas of an equilateral triangle, square, pentagon, hexagon, and octagon, inscribed in a circle of unit radius.

Chapter II.

Motion. *Velocity.*

13. *Motion* is change of position. Although the ideas conveyed by the terms matter and motion are quite different, yet it is evident that all the motions we are cognizant of are the motions of matter directly, or are indirectly produced by motions of matter. Thus the motion we see when a boy throws a stone is the motion of the stone directly. A *wave*, on the other hand, which is *motion of form*, is not directly the motion of the medium through which the wave is passing, but is indirectly produced by the motion of this medium.

14. The opposite (or the zero) of motion is *rest.* All the motion or rest of a body that we can know of is *relative*, *i.e.* with respect to some other body. In infinite space absolute motion or rest is indeterminable if indeed conceivable. When a person is sitting at his ease in a railway carriage, he is said to be at rest. But this is merely *relatively* to the train. Relatively to the earth he is moving as fast as the train is, and when we consider that the earth is rotating about its axis, is further revolving around the sun, and with the sun and other members of the solar system careering through space, it is easily seen how complex is the person's motion. The aim of the physicist is to determine those conditions of matter and motion which, apart from the world of sensation, thought, and consciousness, constitute the life of the universe.

15. *Velocity* is rate of motion, *i.e.* rate of change of position per unit of time. By *rate* is meant here degree of quickness. When two bodies are moving, and one moves

over a greater distance in the same time than the other, the velocity of the former is said to be the greater. In any motion of a body the velocity may be *uniform*, *i.e.* the same throughout the motion, or it may be *variable*, *i.e.* continuously or at intervals changing during the motion. The velocities of all bodies that we see moving are really variable. The motions of the hands of a chronometer, or the rotations about their axes of the different members of the solar system are cases of motion in which the velocities are nearly uniform. The test of uniform velocity is that equal distances are moved over in equal times, *however small these times may be.*

16. What, however, we naturally ask, are *equal times?* We look at our clocks or watches and say that they tell us equal times. Some watches go slow, others go fast, and how are we to know which go right? It is well-known that our clocks and watches are regulated by the apparent motion of the sun in the sphere of the heavens. This motion is the resultant of two motions, viz., (1) the apparent rotation of the sphere of the heavens, produced by the real rotation of the earth, which takes place in a sidereal day; and (2) the apparent revolution of the sun in the ecliptic in a sidereal year, produced by the real revolution of the earth in its orbit around the sun. As the sidereal year is however estimated in sidereal days, we find that ultimately the apparent rotation of the sphere of the heavens is our standard measurer of time, and we define *equal times* as times in which the sphere of the heavens apparently rotates through equal angles. Whether the term equality is rightly applied to such times or not is a legitimate enquiry.

17. The mean or average time in which the sun apparently rotates about the earth is called a mean solar day, and

our unit of time, the *second*, is a well-known fraction of the mean solar day. In the scientific system of units of measurement the *second* is, like the unit of length, one of the fundamental units.

18. The magnitude of a uniform velocity is measured by the number of units of length passed over in a unit of time. The unit of velocity is derived immediately from the units of length and time; it is the velocity in which a unit of length is passed over in a unit of time. Hence the scientific unit of velocity is 1 centimetre per second. It will be convenient to call this a *tach*. If s be the distance in centimetres described in t seconds by a body moving uniformly with a velocity of v tachs, then $s = vt$, and $v = \frac{s}{t}$.

19. When a body is moving with variable velocity it has of course a definite velocity at every instant, which is measured by the number of units of length which *would be* passed over in a unit of time, if for such a period from the instant in question the velocity did not change. Hence we talk of a ship sailing at the rate of 12 knots an hour, of a man walking at the rate of 4 miles an hour, &c., although the velocity of the ship and of the man may not be the same for any two consecutive seconds. When a body is moving with variable velocity, the equation $v = \frac{s}{t}$ gives the mean velocity during the time t, and, by taking t small enough, we can approximate in any degree of exactitude to the velocity at any instant.

20. A velocity has direction as well as magnitude, and can be completely represented by a straight line, the direction of the line representing the direction of the motion (the tangent to the path of the moving particle at the

instant in question), and the length of the line representing the magnitude of the velocity. Since a body may move in two directions along a line, the one being diametrically opposite to the other, it is convenient to distinguish these by the signs + and −, as is customary in the applications of algebra to geometry. If AB be a straight line, and a velocity in the direction of AB be called +, a velocity in the direction of BA will be called −.

Angular Velocity.

21. The only case of angular velocity we need at present consider is that about the centre of a circle of a body moving in the circumference.

Definition. When a body moves in a circle the rate of change of the angle, which the radius through the body makes with a fixed radius, is called the angular velocity of the body about the centre of the circle.

If the angles described by the radius through the body be equal in equal times, however small these may be, then the angular velocity is uniform and is measured by the angle described in a unit of time. The unit of angular velocity is that in which a unit of angle is described in a unit of time. In scientific measure, then, the unit of angular velocity will be a radian per second.

When the body is revolving with variable velocity, the angular velocity at any instant is measured by the angle which *would be* described by the radius through the body in a unit of time, if for such a period the velocity did not change.

22. From the formula $a = r\theta$ (art. 9) it follows at once that if v represent in tachs the linear velocity of a body,

moving in the circumference of a circle, r the radius in centimetres, and ω the angular velocity about the centre in radians per second,

$$v = r\omega, \text{ and } \omega = \frac{v}{r}.$$

23. If the angular velocity be in the direction of the hands of a watch it is represented by the $-$ sign, and if in the opposite direction by the $+$ sign. Let it be carefully observed, however, that the sign given to the angular velocity of a body revolving in a circle depends upon the side of the plane of motion from which the motion is observed. Thus, if we could see the motion of the hands of a watch through the back of the watch, the angular velocity would be $+$. If we look northward at the rotation of the sphere of the heavens it seems to be $+$, and if we look southwards it seems to be $-$.

Examination II.

1. Define motion, rest, velocity.

2. What is a wave? How is it produced? Give examples.

3. Illustrate the meanings of the terms relative and absolute, (1) with respect to extension, (2) with respect to motion, (3) with respect to direction.

4. Define velocity, and distinguish between uniform and variable velocity.

5. What is the test of uniform velocity?

6. Define *equal times*, and the unit of time.

7. Distinguish between fundamental and derived units, and give examples of each.

8. Name and define the scientific unit of velocity.

9. Give the relation between s, v, t in uniform motion.

10. When the motion of a body is not uniform, how is the velocity measured?

11. How may velocity be completely represented?

12. Define angular velocity, and the unit thereof.

13. Give and prove the relation between the angular and linear velocity of a body moving in the circumference of a circle.

14. Distinguish between + and – angular velocity.

Exercise II.

1. A body has a velocity of 10 tachs, how long will it take to pass over 600 metres?

2. Which is greater a velocity of 72 tachs or one of 252 metres per hour, and by how much?

3. Express a velocity of 72 kilometres per hour in decimetres per minute.

4. If a line a foot long represent a velocity of 3·75 miles per hour, what length of line would represent a velocity of 80 yards per minute?

5. Two bodies start from the same point, the one 10 minutes after the other, and travel in perpendicular directions with velocities of 120 tachs and 100 metres per minute. How far apart will they be in an hour from the starting of the first?

6. Two travellers leave the same place at the same time

in directions inclined to one another at an angle of $\frac{\pi}{3}$, and each travels with a velocity of 166 tachs, how far apart will they be in two hours?

7. A man two metres high walks in a straight line at the rate of 6 kilometres an hour away from a lighted lamp 3 metres high; find in tachs the velocity of the end of his shadow, and the rate at which his shadow lengthens.

8. If 3 minutes be the unit of time, and 50 decimetres the unit of length, what number measures the average rate of walking of a person who goes over 40 kilometres in 12 hours?

9. If 7 metres per 3 minutes be the unit of velocity, and 4 decimetres the unit of length, what must be the unit of time?

10. If 3 metres per 7 minutes be the unit of velocity, and 4 seconds the unit of time, what must be the unit of length?

11. A body moving uniformly in a circle describes the circumference twice in 3 minutes, what is the scientific measure of its angular velocity?

12. What is the angular velocity of any body on the earth's surface due to the earth's rotation?

13. The diameter of the driving wheel of a locomotive is 2 metres, what is the angular velocity of a point on the circumference when the train is moving at the rate of 80 kilometres an hour?

14. A body moving in the circumference of a circle of radius 10 has unit angular velocity; find the space described in 10 seconds, and the time taken to complete a revolution.

Chapter III.

Acceleration

24. Just as a body's position may be constantly changing, giving rise to motion, so a body's velocity may be constantly changing, giving rise to acceleration.

Acceleration is change of velocity, and is measured by the rate of this change per unit of time. A body's velocity may change in magnitude only, or in direction only, or in both magnitude and direction, and in any one of these cases it will be found that at every instant the body has a definite acceleration.

The rate at which a body's velocity changes may be slow or fast. Compare the accelerations of trains in a long railway like the Canadian Pacific, in which the stations are far apart, with the acceleration of trains in large cities, run for the convenience of passengers hurrying from one part of the city to another, such as the Underground Railroad in London or the Elevated Railroad in New York.

25. Acceleration, as change of *velocity*, has direction as well as magnitude, and the direction of a body's acceleration at any instant may, or may not, be the same as the direction of its motion.

Let us first consider acceleration, the direction of which is the same as that of the body's motion, or diametrically opposite thereto. The effect of such an acceleration is evidently to change the magnitude of a body's velocity without changing its direction. Acceleration may be uniform or variable. An acceleration is uniform when equal changes of velocity take place in equal times, however small these times may be, and is then measured by the

number of units of velocity acquired in unit of time. Thus, if in every unit of time the body's velocity is changed by a units of velocity, then a measures the magnitude of the acceleration. If the velocity is increased by a, the acceleration is generally represented by $+a$; if on the other hand the velocity be diminished by a units of velocity in unit of time, the acceleration is represented by $-a$.

26. The unit of acceleration is a change of unit of velocity in unit of time; hence the scientific unit of acceleration is 1 tach per second.

Observe carefully that the unit of acceleration, by involving the units of velocity and time, involves the unit of length once and the unit of time twice. This must be particularly attended to if in the solution of a problem the units require to be changed. Thus a tach is represented by $\frac{60}{100}$ if a metre and minute be the units of length and time, but with the same units the scientific unit of acceleration will be represented by $\frac{60 \cdot 60}{100}$. One of the most important cases of the motion we are now considering is that of a body moving vertically upwards or downwards *in vacuo.* Such a body has a uniform acceleration vertically downwards. This is represented by g, and in the latitude of Kingston, Ont., its value is 980·5, *i.e.* in every second the body acquires a downward velocity of 980·5 tachs.

If a represent the acceleration of a body uniformly accelerated in the direction of its motion ($+\,ly$ or $-\,ly$), and v be the whole change of velocity in time t, then $v = at$ and $a = \dfrac{v}{t}$.

27. A body may have an acceleration constant in magnitude and direction, but different in direction from the

direction of motion. The resultant motion in this case is very different from that of the preceding case. Such would be the motion of a body moving *in vacuo* in any but a vertical direction; it is very nearly that of a leaden bullet projected in the air in any but a vertical direction, and with a small velocity. The magnitude of such a body's velocity will be constantly though not uniformly changing, and the direction of motion will be constantly changing, so that the path described will be a parabola with its axis in the direction of acceleration.

28. Again, a body may have an acceleration constant in magnitude but not in direction. Any body revolving uniformly in a circle (which is approximately the motion of the moon in its orbit) has such an acceleration. If ω be the angular velocity of the body and r the radius of the circle, it can be shewn that the magnitude of acceleration is measured by $r\omega^2$, but the direction of acceleration is always towards the centre of the circle and therefore changing at every instant.

29. When the magnitude of a body's acceleration is variable, the acceleration at any instant is measured by the number of units of velocity by which the body's velocity *would be* changed in a unit of time, if for such a period from the instant in question the acceleration did not change.

When the acceleration is variable the formula $a = \frac{v}{t}$ gives the average acceleration during the time t, and by taking t small enough we may approximate as closely as we please to the acceleration at the beginning of time t.

30. A cannon ball shot vertically upwards with great velocity is an example of a body, whose acceleration is constant in direction, but variable in magnitude on account of

the varying resistance of the air. If the ball be shot in any but a vertical direction we have a case of motion in which the acceleration is constantly changing both in magnitude and direction.

31. Acceleration, like velocity, is completely represented by a straight line, the direction of the line being the direction of acceleration, and the length of the line representing the magnitude of the acceleration.

Examination III.

1. Define acceleration. How is it measured?

2. Define the unit of acceleration? What fundamental units does it involve?

3. What is the acceleration of a falling body in the latitude of Kingston, Ont.? What of a body rising vertically upwards?

4. Under what conditions is the formula $v = at$ true?

5. How is acceleration measured when variable?

6. Give examples of bodies having accelerations, (1) constant both in magnitude and direction, (2) constant in magnitude but not in direction, (3) constant in direction but not in magnitude, (4) variable both in magnitude and direction.

7. Shew that an acceleration of a metre per minute per second is equal to an acceleration of a metre per second per minute.

Exercise III.

N.B.—In the following examples the acceleration is supposed to be uniform and in the direction of motion.

1. A body has an acceleration 20; find in decimetres per minute the velocity acquired in an hour.

2. Express the acceleration of a body falling in vacuo (980·5) in units of a metre and an hour.

3. The acceleration due to the weight of a body in feet and seconds is 32·2; find the same in yards and hours.

4. A body is thrown vertically upwards with a velocity of 6000 tachs; what is its velocity at the end of 4 and of 8 seconds, neglecting the resistance of the air?

5. A body uniformly accelerated starts with a velocity of 6 metres per minute, and in half an hour has a velocity of 36 kilometres per hour; find the acceleration in tachs per second.

6. Compare the acceleration 2 with that in which a velocity of 1800 metres per hour is acquired in an hour.

7. Compare an acceleration 3 when a yard and minute are the units with an acceleration 1 when a foot and second are the units.

8. If 1 tach per 10 seconds were the unit of acceleration, what would be the measure of an acceleration of 10 tachs per second?

9. If 6 kilometres per second per minute were the unit of acceleration, and 1 metre the unit of length, what would be the unit of time?

10. If 1 decimetre per hour per second were the unit of acceleration, and 1 metre per minute the unit of velocity, what would be the units of length and time?

11. What is the difference between an acceleration of a metre per hour per second and one of a metre per minute per minute?

12. Find in tachs per second the difference between an

acceleration of 24 metres per minute per second and one of $21\frac{3}{5}$ kilometres per minute per hour.

13. If 216 kilometres per minute per hour be the unit of acceleration, and a second be the unit of time, what must be the unit of length?

14. If the unit of velocity be 96 metres per 15 minutes and 10 seconds be the unit of time, express in scientific measure the unit of acceleration.

15. If the unit of velocity be 5 tachs and 3 metres be the unit of length, express in kilometres per hour per hour the unit of acceleration.

16. If 7 metres be the unit of length and 3 minutes the unit of time, what velocity in tachs will a body acquire in half an hour with an acceleration 18?

17. A body starts with a velocity of 120 tachs, and has an acceleration of 6 metres per minute per minute; another starts at the same time from rest with an acceleration of 36 kilometres per hour per hour; when will their velocities be equal?

18. If 5 inches represent an acceleration of 10 tachs per minute, what length of line will represent an acceleration of 6 when a metre and minute are the units of length and time?

19. If a represent an acceleration when m seconds and n feet are the units of length and time, and a' when n seconds and m feet are the units, find the ratio of $\frac{a}{a'}$ in terms of m and n.

20. If a and a' represent the same acceleration according to two sets of units, and r be the ratio of the units of length, and s the ratio of the units of time, shew that $a : a' :: s^2 : r$.

21. A body is thrown vertically upwards with a velocity of 10 kilotachs; after how many seconds will it be moving downwards with a velocity of 5 kilotachs?

Chapter IV.

Uniformly accelerated Motion.

32. We shall in this chapter consider more fully the nature of the motion of a body which is uniformly accelerated in the direction of its motion. If u be the velocity at any instant and a the acceleration, the velocity at the end of any time t will evidently be $u + at$; denoting this by v we get the first equation of motion,

$$v = u + at.$$

33. *To determine the space described in the time t.* Since the velocity during the time t increases *uniformly* from u to $u+at$, the *average velocity* is $\frac{1}{2}\{u+(u+at)\}$ or $u+\frac{1}{2}at$. If a body moved *uniformly* during the time t with this velocity, the space described would be $(u+\frac{1}{2}at)t$ or $ut+\frac{1}{2}at^2$. This will evidently be also the space described during the time t by a body whose velocity increases *uniformly* from u to $u+at$ in that time. Hence if a body has an initial velocity u, and an acceleration a in the direction of its motion, and if s denote the distance described in the time t,

$$s = ut + \tfrac{1}{2}at^2.$$

34. The following may be considered by the student a more rigorous proof of the same result:

Let the time t be divided into *any* large number of equal parts. If n denote the number, the duration of each little interval will be $\frac{t}{n}$. The velocities at the *beginning* of each of the little intervals will be

$$u\,,\ u+a\frac{t}{n}\,,\ u+2a\frac{t}{n}\,,\ \ldots\ldots\ldots\ldots u+(n-1)a\,\frac{t}{n}.$$

The velocities at the *end* of each of the little intervals will be

$$u+a\frac{t}{n}\ ,\ u+2a\frac{t}{n}\ ,\ u+3a\frac{t}{n}\ ,\ \ldots\ldots\ldots\ldots u+na\frac{t}{n}.$$

Suppose now that a body A moved *uniformly* during each little interval with the velocity indicated above at the beginning of each interval; if s_1 be the whole distance moved over during the n intervals, *i.e.* during the time t,

$$s_1=\frac{t}{n}\left\{u+(u+a\frac{t}{n})+(u+2a\frac{t}{n})+\ldots(u+\overline{n-1}a\frac{t}{n})\right\}$$

$$=ut+a\frac{t^2}{n^2}\left(1+2+3+\ldots\ldots\ \ldots\ldots\ldots\overline{n-1}\right)$$

$$=ut+a\frac{t^2}{n^2}\cdot\frac{n(n-1)}{2}=ut+\tfrac{1}{2}at^2\left(1-\frac{1}{n}\right)$$

Similarly if s_2 be the whole distance moved over by a body B, which moved uniformly during each little interval with the velocity indicated above at the end of each interval,

$$s_2=ut+\tfrac{1}{2}at^2\left(1+\frac{1}{n}\right)$$

Now if s be the whole space described during the time t by the body uniformly accelerated, it is evident that s is greater than s_1 and less than s_2, for the body A moved during each little interval with the *least* velocity, which the body uniformly accelerated had during that interval, and the body B with the *greatest*.

$$\therefore\quad s>ut+\tfrac{1}{2}at^2\left(1-\frac{1}{n}\right)$$

$$<ut+\tfrac{1}{2}at^2\left(1+\frac{1}{n}\right)$$

Now these two quantities between which s lies differ only in the sign of $\frac{1}{n}$. What is n? n is *any number whatsoever*, and may be made as large as you please. But by taking n large enough, $1-\frac{1}{n}$ and $1+\frac{1}{n}$ may be made to differ from 1 by as small a fraction as you please. Hence when n becomes *indefinitely* great, the motions of A and B do not differ from the motion of the body uniformly accelerated, and the three quantities s, s_1, s_2 become $ut+\frac{1}{2}at^2$.

35. From the equations of uniformly accelerated motion just determined

$$v = u + at \quad \ldots\ldots (1)$$
$$s = ut + \tfrac{1}{2}at^2 \quad \ldots\ldots (2)$$

we derive by algebraical analysis the following useful though not independent equation:

$$v^2 = u^2 + 2as \quad \ldots\ldots (3)$$

Cor. 1. If the acceleration be opposite in direction to that of motion, it must be represented by $-a$, and the equations become

$$v = u - at \quad \ldots\ldots (4)$$
$$s = ut - \tfrac{1}{2}at^2 \quad \ldots\ldots (5)$$
$$v^2 = u^2 - 2as \quad \ldots\ldots (6)$$

Cor. 2. If the body start from rest, $u=0$ and the equations become

$$v = at \quad \ldots\ldots (7)$$
$$s = \tfrac{1}{2}at^2 \quad \ldots\ldots (8)$$
$$v^2 = 2as \quad \ldots\ldots (9)$$

Comparing (2) or (5) with (8) we might say that ut is

the space described in virtue of the velocity u, and $\frac{1}{2}at^2$ that described in virtue of the acceleration a.

36. As already stated in art. 26, the motion of a body moving freely in a vertical direction is of the character we have been considering. Strictly speaking, this applies only to bodies moving *in vacuo*, but unless the velocity be great we may often neglect the action of the atmosphere. The acceleration g of such bodies is vertically downwards, and in the latitude of Kingston, Ont. is very nearly 980·5 tachs per second, or 32·17 feet per second per second.

Let us consider the motion of a body thrown vertically upwards with a velocity u.

1). *How long will it rise?*

It rises until its velocity is zero. Hence from equation (4) we get $0 = u - gt$, $\therefore\ t = \dfrac{u}{g}$.

2). *What is the greatest height reached?*

In equation (5) putting $t = \dfrac{u}{g}$ we get

$$s = u\frac{u}{g} - \frac{1}{2}g\left(\frac{u}{g}\right)^2 = \frac{u^2}{2g}.$$

We might get the same result more simply from equation (6). When the body ceases to rise, $v = 0$

$$\therefore\quad 0 = u^2 - 2gs, \qquad \therefore\quad s = \frac{u^2}{2g}.$$

3). *When will the body return to the point of projection?*

The distance described *from the point of projection* in the required time is zero; hence from equation (5),

$$0 = ut - \frac{1}{2}gt^2, \qquad \therefore\quad t = 0 \text{ or } \frac{2u}{g}.$$

Comparing this result with 1), we see that the time taken for a body to fall from the greatest height reached, back again to the point of projection, is just the same as that taken by the body to reach its greatest height.

4). *What is the velocity of the body after returning to the point of projection?*

$$\text{From equation (4), } v = u - g\left(\frac{2u}{g}\right) = -u,$$

that is, the velocity is the same in magnitude as that at starting, but opposite in direction. Now, *since any point in the path might be considered a point of projection*, we infer from this result that the return or downward motion of the body is an *exact image* of the upward motion.

37. *The distances described in successive seconds (or other equal intervals of time) by a body, which starts from rest and is uniformly accelerated, are as the odd numbers.*

The distances described in 1, 2, 3, ... $(n-1)$, n seconds are $\frac{1}{2}a(1)^2$, $\frac{1}{2}a(2)^2$, $\frac{1}{2}a(3)^2$ ... . $\frac{1}{2}a(n-1)^2$, $\frac{1}{2}an^2$; $\therefore$ the distances described in the 1st, 2nd, 3rd, ...nth seconds are $\frac{1}{2}a$, $\frac{3}{2}a$, $\frac{5}{2}a$, ...$\frac{1}{2}a(2n-1)$. Thus the distance described in the nth second, *where n is any number whatsoever*, is equal to $\frac{1}{2}a(2n-1)$, which varies as $(2n-1)$ the nth odd number.

Ex. A body is thrown vertically upwards with a velocity of 3922 tachs; find 1) the time taken to describe 58·83 metres, 2) the velocity at that height, 3) the greatest height reached, 4) the time of ascent, 5) the distance described in the half second following the fifth second from the instant of starting, 6) the distance described in 10 seconds.

1). Let t = time required. From equation (5),

$$5883 = 3922\,t - \tfrac{1}{2}(980{\cdot}5)t^2, \quad \therefore t = 2 \text{ or } 6 \text{ seconds}.$$

2). From equation (4), velocity at the end of 2 seconds $=3922-2\,(980{\cdot}5)=1961$ tachs; velocity at the end of 6 seconds $=3922-6\,(980{\cdot}5)=-1961$ tachs.

We thus see from 1) and 2) that the body in its *ascent* has risen 58·83 metres in 2 seconds; that after 6 seconds it is at the same height, but is then *descending;* that at both times the *magnitude* of the velocity is the same.

3). From 2), art. 36, the greatest height reached

$$=\frac{(3922)^2}{980{\cdot}5} = 15688 \text{ centimetres.}$$

4). From 1), art. 36, the time of ascent $= \dfrac{3922}{980{\cdot}5} = 4$ sec.

5). Distance described in $5\frac{1}{2}$ seconds

$$= 3922\left(\tfrac{11}{2}\right) - \tfrac{1}{2}(980{\cdot}5)\times\left(\tfrac{11}{2}\right)^2$$

Distance described in 5 seconds $=3922\times5-\frac{1}{2}(980{\cdot}5)\times5^2$

$\therefore$ the required distance

$$=3922\times\tfrac{1}{2}-\tfrac{1}{2}(980{\cdot}5)\times\tfrac{21}{4} = -612\tfrac{13}{16} \text{ cm.}$$

The $-$ sign tells us that the body has *descended* down this distance in the 11th half second of its motion.

6). From equation (5) the distance required is

$$3922\times10-\tfrac{1}{2}(980{\cdot}5)\times10^2 = -9805 \text{ cm.}$$

The $-$ sign tells us that the body is *below* the point of projection.

Examination IV.

1. Determine the equations of motion of a body uniformly accelerated in the direction of its motion.

2. Deduce the formula $v^2=u^2-2as$.

3. A body starts from a given point with a velocity u, and has an acceleration a opposite in direction to u; determine 1) after what time will the velocity be zero? 2) After what time will the body return to the point of projection? 3) What is the velocity on returning to the point of projection? 4) What is the greatest distance travelled over?

4. Give the three equations of motion of a body let fall to the ground, neglecting the resistance of the air.

5. Prove that the distances described in successive equal intervals of time by a body, which starts from rest and is uniformly accelerated, are as the odd numbers.

6. Trace the motion of a body projected vertically upwards, and shew that the downward return motion is an exact image of the upward.

Exercise IV.

N.B.—In the following examples the directions of velocity and acceleration are the same, and in the case of bodies moving vertically the resistance of the air is neglected, unless otherwise stated.

1. A stone is observed to fall to the bottom of a precipice in 9 seconds; what is the depth? Given $g = 980$.

2. The height of the piers of Brooklyn Bridge is 277 ft.; how long will a stone let fall from the top take to fall into the water? Given $g = 32\frac{1}{6}$.

3. A body is projected vertically upwards with a velocity of 320 ft. per sec. 1). How long will it rise? 2). How far will it rise? 3). When and where will its velocity be 150 miles per hour? 4). How long will it take to rise 1000

ft.? 5). What will its velocity be at that height? 6). How far will it travel in the seventh second? Given $g=32$.

4. A body starts with a velocity of 1 metre per second, and has an acceleration of 10 tachs per second; what will its velocity be after traversing $6\frac{1}{4}$ metres?

5. How long would a body in Kingston, Ont., which is projected with a downward velocity of 450 tachs, take to fall through 15 kilometres, if there were no atmospheric resistance?

6. The velocity of sound in air is uniform and at 10 °C. is equal to 33833 tachs. The depth of the well in the fortress of Königstein in Saxony is 195 metres. In what time should the splash of a stone dropped into the well be heard, if there were no atmospheric resistance?

7. When a bucket of water is poured into this well, the splash is heard in 15 seconds; what is the *average* acceleration produced in the water by the resistance of the air?

8. A body whose acceleration is 10, traverses 6 metres in 10 seconds; what is the initial velocity?

9. A body moves over 34·3 metres in the fourth second of its motion from rest; find the acceleration.

10. A person, starting with a velocity of 1 metre per second, and accelerating his speed uniformly, traverses 960 metres in a minute; find his acceleration.

11. A body starts from a given point with a uniform velocity of 9 kilometres per hour; in an hour afterwards another body starts in pursuit of the first with a velocity of 2 metres per second, and an acceleration of 5 decatachs per hour; when and where will the second body overtake the first?

12. A body projected vertically upwards in Kingston, Ont., passes a point 10 metres above the point of projection with a velocity of 9805 tachs; how high will it still rise, and what will be its velocity on returning to the point of projection?

13. A body uniformly accelerated describes 6·5 metres and 4·5 metres in the fourth and sixth seconds of its motion; find the initial velocity and acceleration.

14. Two bodies uniformly accelerated in passing over the same space have their velocities increased from a to b, and from c to d respectively; compare their accelerations.

15. Find the acceleration when in one-tenth of a second a velocity is produced, which would carry a body over 10 metres every tenth of a second.

16. A particle is projected vertically upwards, and the time between its leaving a point 21 feet above the point of projection and returning to it again is observed to be 10 seconds; find the initial velocity. Given $g=32$.

17. Two bodies are let fall from the same place in Kingston, Ont. at an interval of two seconds; find their distance from one another at the end of five seconds from the instant at which the first was allowed to fall.

18. Two bodies let fall from heights of 40 metres and 169 decimetres reach the ground simultaneously; find the interval between their starting. Given $g=980$.

19. Two bodies start from rest and from the same point on the circumference of a circle; the one body moves along the circumference with uniform velocity, and the other, starting at the same time, moves along a diameter with uniform acceleration; they meet at the other extremity of the diameter; compare their velocities at that point.

20. A body, starting from rest with an acceleration of 20 tachs per second, moves over 10 metres; find the whole time of motion, and the distance passed over in the last second.

21. A body moves over 9 ft. whilst its velocity increases uniformly from 8 to 10 ft. per second; how much farther will the body move before it acquires a velocity of 12 ft. per second?

22. The path of a body uniformly accelerated is divided into a number of equal spaces. Shew that, if the times of describing these spaces be in $A. P.$, the mean velocities for each of the spaces are in $H. P.$

23. A body falling freely is observed to describe $24\frac{1}{2}$ metres in a certain second; how long previously to this has it been falling?

24. A body is dropped from a height of 80 metres; at the same instant another body is started from the ground upwards so as to meet the former half way; find the initial velocity of the latter body, and the velocities of the two bodies when they meet.

25. A body has a uniform acceleration a. If p be the mean velocity, and q the change of velocity, in passing over any portion s of the path, shew that $pq = as$.

26. A body uniformly accelerated is observed to move over a and b feet respectively in two consecutive seconds; find the acceleration.

CHAPTER V.

Inertia. *Mass.*

38. *Inertia* is the inability of a body in itself to alter its own condition of motion or rest. If a body be at rest, it remains so; if it be in motion, it goes on moving in the same direction and with the same velocity, *i.e.* uniformly in a straight line; and if it be rotating, it goes on rotating with the same angular velocity, about the same axis, which maintains a constant direction; *unless some other body interfere with it. To change the state of rest or motion of a body requires the presence of another body. Force* is the term applied to the action of a body in altering the *status quo* of another body.

39. Inertia, although a *negative* property, is perhaps the most obtrusive property of matter. It is lucidly illustrated in railway and horse-riding accidents, in vaulting and jumping, in shaking the dust from off a book, in the danger of turning round a corner in a carriage very rapidly, in the difficulty of driving over smooth ice, and in the action of a fly-wheel, which is used to regulate either an irregular driving power, as in a foot-lathe, or an irregular resistance, as in a circular saw cutting wood. The tendency of bodies, moving in circles, to fly off at every instant along the tangent, commonly but misleadingly called *centrifugal force*, is just a case of inertia. Herein we have an explanation of the spheroidal form of the earth, and of the diminution of a body's weight, as we approach the equator. On letting a bullet fall from the top of a high tower or down a deep mine, it will, on account of its inertia, be found to fall somewhat to the *east* of the point vertically below that from which it fell, thus affording an ocular demonstration of the

earth's rotation from *west* to *east*. The rotations of the earth and other members of the solar system afford beautiful examples of the inertia of rigid bodies as regards rotation. The *constancy of direction* of the earth's *axis*, (except in so far as it is interfered with by the sun and moon), furnishes us with the most important step in the explanation of the changes of the seasons. The *gyroscope* is a beautiful physical toy which illustrates the same important principle. In Foucault's experiment for proving the earth's rotation, the same principle is assumed for an oscillating pendulum as regards its plane of oscillation.

40. Newton clearly enunciated the Inertia of Matter in his First Law of Motion :

Every body continues in its state of rest, or of uniform motion in a straight line, except in so far as it may be compelled by impressed force to change that state.

Observe, however, that he takes no notice of inertia as regards *rotation*. Here indeed there is a difficulty, for evidently the individual small particles of the rotating body move in circles, and must therefore be acted on by forces among themselves; else, on account of their inertia, they would move uniformly in straight lines. However, when once by *internal* forces the relative positions of the particles are fixed, the body will be as inert in its rotation, as in its motion of translation.

41. When the *same* force acts on *different* bodies we find that the changes from the previous states of rest or motion are different, and we express this fact by saying that the bodies differ in *mass*. *Mass*, then, is a property in which bodies differ, just as they differ in colour, volume, elasticity, &c. It may not inaptly be called the *dynamical measure of a body's inertia.*

How is Mass measured?

When the same force acting during the same time on two bodies produces in each the same changes of velocity, the masses are defined as equal to one another; but, if the changes of velocity be not the same, the masses are defined as inversely proportional to the changes of velocity produced.

42. The difference in mass of different bodies (*e.g.* of balls of wood, ivory, lead, &c., of different radii) may be lucidly illustrated by suspending the bodies by strings, and allowing the same spring, bent through the same angle, to act upon them in succession so as to give the bodies a horizontal motion. It will be found that the velocities imparted will be very different.

43. A unit volume of pure water at 4° C. (and under the mean atmospheric pressure) is defined as having *unit of mass*, which is called a *gram.* In the scientific system of the units of measurement the gram is the third and last of the *fundamental units.* The English standard unit of mass is the *pound avoirdupois*, which contains 7000 grains. The French standard unit of mass is the *kilogramme*, which is the mass of a litre of pure water at 4° C. What have hitherto been called the scientific units are better called the C. G. S. units, these three letters being the initial letters of the fundamental units adopted by scientific writers. These units, it is agreed, will be used in all international scientific questions. With other fundamental units equally scientific systems can be formed. Thus with the English *foot*, *pound*, and *second* an English scientific system of units is formed, which we shall call the F. P. S. system. It will be useful to the student to familiarize himself with both of these

systems of units in the solution of problems. *Whenever special units are not mentioned, the units of the C. G. S. system are to be understood.*

44. Let it be observed that by means of *one* force we can theoretically determine the masses of *all* bodies.

When the same force acts upon bodies of the *same* kind, *e.g.* two pieces of iron at the same temperature, it is found that the accelerations produced are inversely proportional to their volumes, (take as an illustration the opening of doors of the same kind of wood but of different sizes); but, when the same force acts upon bodies of *different* material, *e.g.* a piece of iron and a piece of wood, it is found that the accelerations produced are not inversely proportional to their volumes, (take as an illustration the opening of a wooden and of an iron door). Hence it follows that *the masses of bodies of the same material* (and at the same temperature and pressure) *are directly proportional to their volumes;* but *not so* for bodies of *different* material.

45. These facts lead us to the consideration of *specific mass*, or, as it is more commonly called, *density*. The *density* of a body is *the mass per unit of volume.* Hence the density of water at 4° C. will be represented by 1. A unit volume of gold at 0° C. and under the mean atmospheric pressure is found to be 19·3 grams, of rock-crystal 2·65, of cork 0·24, of mercury 13·596, of dry air 0·001293, &c. We express these facts by saying that the density of gold is 19·3, of rock-crystal 2·65, of cork 0·24, of mercury 13·596, of dry air 0·001293, &c.

46. The density of water, as of all other substances, varies with temperature, and under the mean atmospheric pressure is a maximum at 4° C. Hence it is that, in

defining unit of mass, the water is taken at this temperature. So little is the density of water changed by pressure that it is hardly necessary to state that the water, in defining unit of mass, is supposed to be under the mean atmospheric pressure, the changes of atmospheric pressure making only immeasurably small changes of density.

47. The density of a body may be uniform, *i.e.* every part having the same density, or it may be variable. In the latter case the density at any point of the body is the same as that of any other body, whose density throughout is uniform and the same as at the point in question.

48. We might have defined the density of a body as *the ratio of its mass to the mass of an equal volume of pure water at* $4°$ *C.*

In the case of a body of variable density, this definition would give us the *mean density* of the body. It would further be applicable whatever units of volume and mass be taken.

49. Relation between the *mass*, *volume*, and *density* of a body, $M = VD$, where M is the mass of the body in grams, V its volume in cubic centimetres, and D its density.

When masses are expressed in pounds avoirdupois and volumes in cubic feet, then $M = 62{\cdot}4\ VD$ gives the relation between mass, volume, and density, since a cubic foot of water is 62·4 pounds.

50. The mass of a body is sometimes defined as *the measure of the quantity of matter in it*, or as *the dynamical measure of the quantity of matter in it.* Since we do not know the ultimate nature of matter this can hardly be scientifically correct. We only know the *properties* of

matter, and can only measure its properties. Why then should quantity of matter be measured by one of these properties, *mass*, rather than by any other? We might reason thus: when the same quantity of heat is applied to bodies of the same kind, the changes of temperature produced are *inversely proportional to their volumes;* but when applied to bodies of *different* kinds, the changes of temperature are *not* inversely proportional to their volumes. We express this fact by saying that bodies differ in *thermal capacity*, and we define the thermal capacities as inversely proportional to the changes of temperature produced. Just then as with mass we might define the thermal capacity of a body as the *thermal measure* of the quantity of matter in it. We should then find that the thermal measure and measurement by mass were quite different.

So long as we are dealing with matter of *one* kind there are many ways in which we may measure quantity of matter quite intelligibly, *e g.* by volume, by weighing in the same place, in the case of food by the length of time it will supply nourishment, in the case of fuel by the amount of heat it will give out, or by the amount of oxygen gas necessary for its complete combustion, &c., and all these measurements would be found to agree with one another as well as with the measurement by mass. But when we come to deal with bodies of *different* kinds, none of these measurements will be found to give results consistent with one another.

51. The reason, doubtless, why *mass* is stated to measure the quantity of matter in a body, is that this is the only property of a body which remains invariable through whatsoever changes the body may pass. Thus, whilst by pressure, motion, heat, chemical action, or other agencies, we can

alter the other measurable properties of a body, such as its volume, elasticity, weight, thermal capacity, &c., its mass, through whatever changes the body may pass, remains unchanged. This may be clearly illustrated by many experiments, *e.g.* by dissolving a piece of loaf sugar in tea, by freezing a body of water, by mixing alcohol and water, and, generally, in all chemical combinations. Hence the great law which forms the foundation of chemical science, the *Conservation of Mass*, which asserts that, *through whatever changes matter may pass, the total mass of the universe remains unchanged.*

Examination V.

1. Define inertia, and state the different forms thereof.

2. Give various illustrative examples of inertia.

3. What is centrifugal force? Suggest a better name for it.

4. How may the earth's rotation from west to east be visibly proved?

5. Enunciate Newton's First Law of Motion.

6. Define mass. How is it measured?

7. Describe a simple experiment to shew difference of mass in different bodies.

8. Name and define the units of mass in the C. G. S. and F. P. S. systems of measurement.

9. What relation exists between the volumes and masses of bodies of the same material? How is this proved?

10. Define density in two ways. Give the densities of a few commonly found bodies.

11. Why is water at 4° taken as the standard substance in measuring mass and density?

12. Give the algebraical equation connecting the mass, volume, and density of a body. In the case of a body of variable density how do you express the relation?

13. In the F. P. S. system of units what is the relation between mass, volume, and density?

14. Criticise the usual definition of mass as the measure of the quantity of matter in a body.

15. How did the above definition probably arise?

16. Enunciate the principle of the Conservation of Mass.

Exercise V.

1. A rectangular block of limestone is 2 metres long, 1·5 metre broad, and 1 metre thick. If 2·7 be its density, find its mass.

2. The sides of a canal shelve regularly from top to bottom. The width of a section at the top is 10 metres, at the bottom 5 metres, and the depth is 3 metres. If the canal be filled with water to a depth of 2·5 metres, find the mass of water per mile of length.

3. If the density of common salt is 2·3 and of sea-water 1·026; find the mass and volume of salt obtained in evaporating 100 litres of sea-water.

4. The density of copper is 8·8, of zinc 6·9, and of brass formed from these 8·4; find the quantity of copper in 100 grams of brass.

5. The mass of a sphere of rock-crystal is 400·5, and its radius 3·3; find its density.

6. Find the mass of the earth, supposing it to be a sphere of radius 6371 kilometres, and of mean density 5·67.

7. Equal masses of copper and tin, whose densities are 8·9 and 7·3, are melted together; what would be the density of the alloy if no contraction or expansion took place?

8. When 63 litres of sulphuric acid, whose s.g. is 1.85, is mixed with 24 litres of water, the volume of the mixture is 86 litres; find the mass and density of the mixture.

9. The density of gold is 19·3, and of quartz 2·65; the mass of a nugget of gold-quartz is 350, and its density is 7·4; find the mass of gold in it.

10. The density of sea-water is 1·026, and of salt 2·3; 100 litres of sea-water is frozen, and 200 grams of ice free from salt formed therefrom; what is the density of the residue?

11. What is the density of mercury, if 9 cubic inches have a mass of 4·42 lbs?

12. If 4 lbs. of silver have the same volume as 3 lbs. of brass, compare the densities of silver and brass.

13. If 3 cubic inches of silver have the same mass as 4 cubic inches of brass, what mass of silver will have the same volume as 10 lbs. of brass?

14. Two bodies whose volumes are as 3 : 4 are in mass as 2 : 3; compare their densities.

Chapter VI.

Momentum. *Force.*

52. The *momentum* of a moving body is measured by the product of the numbers which represent its mass and velocity.

The *unit of momentum* is that of unit mass moving with unit of velocity, and therefore in the C. G. S. system of units will be the momentum of a body whose mass is 1 gram moving with a velocity of 1 tach. It thus involves each of the three fundamental units once. It is obvious that with the above unit the simple formula $M_o = Mv$ expresses the relation between the momentum, mass, and velocity of any body.

To experience that property of a body called its *momentum*, let a person bathe close to a waterfall, say 200 ft. high, when he will feel the *drops of water*, which separate from the main mass, strike his body as if they were sharp stones. If he attempted to enter the main mass of falling water he would be roughly thrown on the ground.

53. The rate of change of momentum per unit of time will evidently be measured by the product of the numbers which represent the mass and acceleration of the moving body. It is called the *acceleration of momentum* of the moving body.

54. *Force is that aspect of any external influence exerted on a body which is manifested by change of momentum.* Whenever the momentum of a body changes, a *force* is said to act on the body. The *magnitude* of the force is measured by the rate of change of momentum per unit of

time, *i.e. by the acceleration of momentum*, and its *direction* is *the direction of the change of momentum.*

55. This is what Newton taught in his Second Law of Motion :

Change of momentum is proportional to the impressed force and takes place in the direction of the straight line in which the force acts.

By *impressed* force Newton meant *external* to the body concerned. It is well in defining force to avoid the word *cause*. All that we are aware of is a change of momentum and the word force is conveniently used as a measure of the rate of this change. Under *energy*, one of the most important properties of matter, the student will learn that *force* may be defined as *the rate of expenditure of energy per unit of length.*

56. The *unit of force* is that force which produces unit acceleration of momentum, *i. e.*, in C. G. S. measure, the force which, acting upon a body whose mass is 1 gram, produces in 1 second a velocity of 1 tach. Such a force is called a *dyne*, and in terms of this unit we have the simple relation $F=Ma$, where F is a force in dynes, M the mass in grams of the body on which the force acts, and a the acceleration produced in tachs per second.

57. When the change of momentum produced by a force takes place in an immeasurably short time (*e. g.* when a cricket ball is struck by a bat), it is practically impossible to measure the force in *dynes*. Such a force is called an *impulsive force*, and is measured by *the whole change of momentum* produced, without any account being taken of *time*.

58. Since force has *direction* as well as *magnitude*, it is,

like velocity, acceleration, or momentum, completely represented by a straight line; the direction of the line being the direction of the force, and the length of the line representing the magnitude of the force.

59. *What does the second law of motion really teach us?* 1). It defines the measurement of mass and force. Just as it is theoretically possible to measure *all* masses by means of *one* force, so is it possible to measure the magnitudes of *all* forces theoretically by *one* mass. When *different* forces act upon the *same* body, the magnitudes of the forces are by definition directly proportional to the accelerations produced. It is indeed evident that *the mass of any body* is measured in grams by the reciprocal of the acceleration in tachs per second produced, when a dyne acts upon it; and *any force* is measured in dynes by the acceleration in tachs per second produced, when it acts upon a body whose mass is a gram. 2). It enunciates the important experimental fact, that with whatever force different masses be measured, and with whatever mass different forces be measured, the measurements will always be alike. 3). It asserts that the effect of a force depends in no way upon the motion of the body, and that, when more than one force is acting on the body, each force produces its effect quite independently of the others.

60. Different names are given to different aspects of force, such as *pressure*, *tension*, *attraction*, *weight*, *repulsion*, *resistance*, *friction*, &c.

Pressure is applied to a force which calls up the idea of *pushing*, *i.e.* a force acting between two bodies, already close together, in consequence of which they tend to approach still nearer to one another.

Tension calls up the idea of *pulling*. It is a force acting between two bodies close together, in consequence of which they tend to move away from one another. Hence we speak of the tension of a stretched rope. Both pressure and tension are applied to the elastic force of a gas ; pressure, when attention is drawn to the gas pushing against the sides of the containing vessel ; and tension, when we think of the particles of gas tending to separate from one another, so as to occupy, if possible, a greater space.

Attraction is a term applied to forces exerted between bodies, when there is no *sensible* material medium through which the force is exerted, and in consequence of which the bodies *approach* one another. The force between two unlike magnetic poles is a familiar case of attraction.

Weight, a well known form of attraction, is applied to the force exerted between the earth and any body on its surface. Forces are often conveniently measured by the weights of bodies of standard mass. Thus, when a force of p grams is spoken of, a force equal to the weight of a body whose mass is p grams is meant. It would be better to speak of a force of *p grams-weight.*

Repulsion is a term generally applied to forces between bodies, when there is no *sensible* material medium through which the force is exerted, and in consequence of which the bodies *recede* from one another. The force between two like magnetic poles is a familiar example of a force of repulsion.

Resistance is a term frequently applied to any force *opposing* the motion of a body, *i.e.* producing a *negative* acceleration. One of the most familiar and important of such resistances is the ubiquitous force of *friction*, a term applied to that force which is called into play, when one

body moves or tends to move over the *surface* of another body. It is principally the force of friction which a locomotive is constantly working against in pulling a train along. The resistance which bodies experience in falling through the air is largely the force of friction between the bodies and the aerial particles they rub against.

61. By many writers the science of force is called *Mechanics*, and is divided into the two branches *Statics* and *Dynamics*. There has, however, been a much better nomenclature adopted lately by the best modern writers on Natural Philosophy. By them the science of force is called *Dynamics*, and they divide it into *Statics* and *Kinetics*.

Statics treats of *equilibrium* or the balancing of forces. It is chiefly concerned in determining the relations which must exist amongst a set of forces which keep a body at rest.

Kinetics treats of change of momentum. The investigation of the motions of the Solar System is the grandest problem in Kinetics, and is commonly known as Physical Astronomy.

Kinematics is *the science of motion*, when studied *without any reference to mass.* It forms an appropriate introduction to Kinetics. Chapters II., III., IV. belong to Kinematics.

Mechanics is the science which treats of the construction and uses of *machines*.

Examination VI.

1. How is the momentum of a moving body measured?

2. Define the unit of momentum, and give the relation between the momentum, mass, and velocity of a body.

3. Define and give the measure of acceleration of momentum.

4. Define force and its measure in magnitude and direction.

5. Enunciate Newton's Second Law of Motion.

6. Name and define the unit of force in the C. G. S. system, and give the relation between force, mass, and acceleration.

7. What is an impulsive force? How is it measured?

8. How may a force be completely represented?

9. State in full what the Second Law of Motion teaches us.

10. Define the terms pressure, tension, attraction, weight, repulsion, resistance, friction.

11. What is meant by a force of 10 pounds, or a force of 10 kilograms? Give better expressions for these.

12. Define the terms Dynamics, Statics, Kinetics, Kinematics, Mechanics.

13. How can the property of matter called momentum be illustrated?

Exercise VI.

1. A kilodyne acts upon a body whose mass is 50 grams; find the velocity and distance passed over at the end of 10 seconds.

2. A body, whose mass is 5, has an acceleration 2. At one instant the velocity is 10; what is the momentum a minute afterwards?

3. Find the acceleration produced by a megadyne acting on a solid sphere whose diameter is a metre and density 10.

4. A force of 50 kilodynes acts upon a body which acquires in 10 seconds a velocity of a kilotach; find the mass of the body.

5. The mean radius of the earth is 20902070 feet, its mean density 5·67, its mean distance from the sun $92\frac{1}{4}$ million miles, and the time of its revolution around the sun $365\frac{1}{4}$ days; compare its momentum with that of a train of 1000 tons mass, rushing along at 60 miles an hour.

6. The distance of Jupiter from the sun is 5·2 times that of the earth, its period $4332\frac{1}{2}$ days, its mass 310 times that of the earth; compare the momenta of Jupiter and the earth.

7. A body of 10 grams mass has a uniform acceleration of 1 metre per minute per minute; what force is acting upon it?

8. A body acted on by a uniform force is found to be moving, at the end of the first minute from rest, with a velocity which would carry it through 20 kilometres in the next hour; compare the force with the weight of the body, which would give it an acceleration $g = 980·5$.

9. What is the change of momentum in a minute of a body whose mass is 10 and acceleration 10?

10. Compare the momentum of a man, whose mass is 140 lbs. and latitude that of Kingston, Ont. (44° 13′), arising from the earth's rotation, with that of a steamer of 10000 tons mass sailing at the rate of 15 miles an hour. Radius of the earth = 20902070 ft.

11. A body, acted on by a force of 100 kilodynes, has its velocity increased from 6 to 8 kilometres per hour in passing over 84 metres ; find the mass of the body.

12. A body of 1 kilogram mass is acted on by a uniform force in the direction of its motion, and is found to pass over 9·05 and 8·05 metres in the 10th and 20th seconds of its motion ; find the force acting on it, and its initial velocity.

13. Two bodies, acted on by equal forces, describe the same distance from rest, the one in half the time the other does ; compare their final velocities and momenta.

14. Two bodies of equal mass, uniformly accelerated from rest, describe the same distance, the one in half the time the other does ; compare the forces acting on the bodies.

15. Two balls, one of silver and the other of ivory, whose diameters are as 1 to 2, are acted upon by the same impulsive force ; the velocities produced are as 11 to 15 ; compare the densities of silver and ivory.

16. If a ship be sailing with a uniform velocity, what relation must exist between the driving power and the resistances of the air and water ?

17. The density of lead is 11·4 and of cork 0·24. Two balls of these substances, whose diameters are as 1 to 10 are acted upon the same force during the same time ; compare their momenta and velocities.

Chapter VII.

Weight.

62. *Weight* is the force which acts between the earth and every body on its surface, in virtue of which any body, unless it be supported, falls to the ground. It is also called the *force of gravity*. All bodies at the same place are found to fall invariably in the same direction relatively to the surface of the earth, and bodies falling in contiguous places fall in parallel straight lines. The direction in which a body falls at any place is called the *vertical* direction at that place, and is easily found by means of a *plumb-line*. Any direction at right angles to the vertical is called *horizontal*. The surface of any liquid, at rest relatively to the earth, is a horizontal plane, except at the edges of the vessel containing it.

63. When bodies fall *in vacuo* under the action of their weights, the accelerations are found to be the same for all *at the same place*. Hence we deduce the very important fact: *The weights of bodies at the same place are directly proportional to their masses.*

Let W, w be the weights in dynes of two bodies at the same place; M, m the masses in grams; g the acceleration in tachs per second of each body, when falling under the action of its weight, then

$$W = Mg, \ w = mg, \ \therefore \ W : w :: M : m.$$

64. The following extract from Lucretius, translated in Young's Lectures on Natural Philosophy, shews that the fact that all bodies would fall equally fast *in vacuo* at the same place, was believed in, if not proved, nearly 2000 years ago:

In water or in air when weights descend,
The heavier weights more swiftly downwards tend;
The limpid waves, the gales that gently play,
Yield to the weightier mass a readier way;
But if the weights *in empty space* should fall,
One common swiftness we should find in all.

65. Weight is measured like any other force in dynes. Thus the weight of a body whose mass is 1 gram is g dynes (in the latitude of Kingston, Ont., 980·5 dynes). Forces are often conveniently measured in terms of the weights of bodies of known mass. Thus we read of a force of a kilogram-weight or a force of 10 pounds-weight, and these expressions are generally abbreviated into a force of a kilogram or a force of 10 pounds. The measure of a force in terms of weight is called its *gravitation* measure, that in dynes being called in contradistinction its *absolute* measure. Since g the acceleration due to the force of gravity varies with latitude, it is evident that the gravitation measure of a force has not a definite value, unless the place, at which the body of definite mass is weighed, be given. The dyne on the other hand is *independent of time and place*, and is hence called an absolute unit.

66. In the F. P. S. system the unit of force is that force which, acting on a body whose mass is 1 pound for a second, generates a velocity of a foot per second, and is known as a *poundal*. Evidently a pound-weight is equal to g poundals, g being now measured in units of a foot and second ($32\frac{1}{6}$ very nearly in the latitude of Kingston, Ont.)

67. Let us now measure in absolute units the mean pressure of the atmosphere. This, like any other fluid pressure, is measured in terms of the force applied per unit

of area. In absolute measure the *unit of fluid pressure*, or of *pressure-intensity* generally, is a pressure of unit of force per unit of area, and will therefore be in C. G. S. units a dyne per square centimetre. In the absence of a better name this unit may be called a *prem*. The mean pressure of the atmosphere (as indicated by the *barometer*) is the same as what would be produced by the weight of a vertical column of mercury 76 cm. long at 0° in the latitude of Paris. The pressure *per sq. cm.* is therefore equal to the weight of 76 cub. cm. of mercury, *i.e.* (since the density of mercury at $0^\circ = 13{\cdot}596$) $76 \times 13{\cdot}596$ or $1033{\cdot}3$ grams-wt. per sq. cm., *i.e.* (since g at Paris $= 980{\cdot}94$) $76 \times 13{\cdot}596 \times 980{\cdot}94$ or $1{\cdot}0136 \times 10^6$ prems.

68. The simple relation between the weights of bodies *at the same place* and their masses gives the best practical method of measuring the masses of bodies, as is done in a common balance. Observe that in a *common balance*, by comparing the weights of bodies with those of *standard* masses, we really measure *mass*; whereas in a *spring balance* we directly measure *weight*.

The law which explains to us the measurement of weight by means of a spring balance is known as Hooke's law: *The extension, compression, or distortion of a solid body, within the limits of elasticity, is directly proportional to the force which produces it.*

69. The direct proportionality between the masses of bodies and their weights at the same place is the probable cause of mass and weight being constantly confounded. The following illustrations in which these two properties of matter are contrasted, will assist the student to apprehend their difference:

1. *a*). The *mass* of a body is the same at whatever part of the earth's surface it be.

b). The *weight* changes with change of place, and is different at the Equator, at either Pole, or at the summit of the Rocky Mountains, from what it is in the class-room.

2. *a*). The opening of a room door is essentially a question of *mass;* and, however heavy the door may be, if the hinges are truly vertical and well oiled, a small child may open it, though slowly.

b). If the same door formed the lid of a box, and swung on horizontal hinges well oiled, the child could not open it unless he had strength enough to exert muscular force equal to at least half the *weight* of the door.

In either case the child has to overcome the force of friction, which, though greater in the first than in the second case, is in either case small.

3. *a*). In moving a cart along a level road the horse has to exert a greater force at starting than afterwards, because he has to exert force to give the *mass* a given velocity, *i.e.* to produce *momentum.* After having started he has only to balance the force of friction.

b). When, however, he comes to a hill he has again to put forth his strength, for now he has, in addition to the force of friction, to overcome part of the *weight* of the cart.

4. *a*). The action of a fly-wheel depends entirely upon its *mass.*

b. The action of a large steel hammer, worked by steam, depends essentially upon its *weight.*

5. *a*). In athletic sports the "long jump" is essentially a question of *mass.*

b). In the "high jump" *weight* in addition has to be considered. Hence the actual distance of the high jump is never so great as that of the long jump.

6. *a*). In an undershot water wheel the miller depends upon the *momentum* (and hence also the *mass*) of the running water to drive the wheel.

b). In an overshot water wheel he depends upon the *weight* of the water which enters the buckets of the wheel.

70. How is g the acceleration due to the force of gravity at any place measured? The most accurate method of finding this important physical constant is by means of pendulum experiments. There is, however, one method of finding a very accurate value of g, which at this stage the student can understand. This is by means of the well-known physical instrument called Attwood's machine. The essential part of the apparatus is a grooved wheel which turns upon an axle, each end of which rests on two other wheels called the *friction wheels*, so that the friction on the axle of the first wheel is reduced to a minimum; over this wheel passes a fine thread connecting two bodies of different weights. If m and m' be their masses, and m be the greater, the bodies will move on account of the greater weight of m with an acceleration equal to $(m - m')g \div (m + m')$ if we neglect friction and the motion of the wheels. This acceleration can evidently be made as small as the experimenter pleases, by making the difference between m and m' small enough. By a clock and suitable adjuncts the acceleration of the moving bodies can be very accurately measured, and therefore g determined.

The following values of g at the sea-level have been determined by experiment and calculation:

	Latitude.	Value of g.
Equator	0° 0′	978·10
Washington	38° 54′	980·08
Kingston, Ont	44° 13′	980·54
Paris	48° 50′	980·94
Greenwich	51° 29′	981·17
Berlin	52° 30′	981·25
Edinburgh	55° 57′	981·54
Pole	90° 0′	983·11

We thus see that the maximum variation is about $\frac{1}{2}$ p.c. of the mean value.

Examination VII.

1. Define weight. By what other name is it known?

2. How is the direction of weight practically found?

3. Define the terms vertical and horizontal, and give an illustration of each.

4. Prove that the weights of bodies at the same place are directly proportional to their masses.

5. Explain what is meant by an absolute unit of force, and express, in absolute units, forces of a pound-weight and of a kilogram-weight.

6. Name and define the absolute unit of pressure-intensity; express, both in gravitation and absolute measure, the mean pressure of the atmosphere.

7. How is mass practically measured?

8. Give illustrations of weight and mass, which contrast with one another, to shew the difference between these two properties of matter.

9. How is the value of g experimentally determined ?

10. Describe the essential parts of Attwood's machine.

11. Give the values of g at the Equator, Kingston, Ont., Paris, and the North Pole, true to a decitach per second.

12. Enunciate Hooke's Law, and apply it to the spring balance.

13. If a merchant buys goods in London by means of a spring balance, and with the same balance sells in Kingston, Ont., will he gain or loose in the transaction ? Why ?

14. Shew that a poundal is nearly equal to the weight of a body whose mass is half an ounce.

Exercise VII.

N.B.—Take g in the following examples equal to 980·5 *or* $32\frac{1}{6}$. *In examples on Attwood's machine friction and the motion of the wheels are to be neglected.*

1. A body whose mass is 10 grams is falling in vacuo ; what is the force acting on it, and its momentum at the end of 10 seconds ?

2. A force of 50 grams-weight acts upon a body which acquires in 10 seconds a velocity of 39·22 tachs ; find the mass of the body.

3. A force produces in a sphere of radius 10 and density 10 an acceleration 100 ; find what weight the force could balance.

4. A body of 3 kilograms mass pulls by its weight another body of 2 kilograms mass along a smooth horizontal plane ; find the momentum at the end of 5 seconds, and the distance passed over.

5. In Attwood's machine, if 10 kilograms be the mass of

one body, and 15 kilograms that of the other; find the acceleration of momentum, and the velocity at the end of two seconds.

6. A force of 10 pounds-weight acts upon a mass of 2 pounds; what is the velocity after traversing a kilometre?

7. A mass of 10 pounds is acted on by a uniform force, and in 4 seconds passes over 200 feet; express in gravitation measure the force acting.

8. A body of 6 lbs. mass pulls by its weight another body of 4 lbs. mass along a smooth horizontal table; find the time taken to move through 965 feet, and the distance described in the last second.

9. In Attwood's machine one mass is known to be 10 lbs., and the distance described in 2 sec. is found to be 16 ft. 1 in.; find the other mass.

10. A mass 5 has an acceleration 2; at one instant the velocity is 10; what is the momentum a minute afterwards? Express in gravitation measure the force acting on the body.

11. Find the unit of time, if a centimetre be the unit of length, a gram the unit of mass, and a gram-weight the unit of force.

12. Find the unit of length, if a second be the unit of time, a gram the unit of mass, and a gram-weight the unit of force.

13. Find the unit of mass, if a second be the unit of time, a centimetre the unit of length, and a gram-weight the unit of force.

14. A sphere of rock-crystal of density 2·68 has a diameter 6·5; find its volume, mass, and weight in Kingston, Ont.

15. The density of sea-water is 1·028; find the pressure in the ocean at the depth of a kilometre.

16. Find in poundals the tensions of the three parts of a string, which supports at different heights bodies of 12, 6, and 4 lbs. mass respectively.

17. Oxygen combines chemically with hydrogen to form steam in the proportion of 8 parts by mass of oxygen to 1 of hydrogen. If the gases be weighed by means of a spring balance graduated at Edinburgh, how many milligrams-weight of oxygen at the Equator will combine with 100 milligrams-weight of hydrogen at Edinburgh to form steam?

18. Determine the mass of steam so formed, and its weight, as indicated on the above spring balance, at Kingston, Ont.

19. Answer the above (18 and 19) when the gases are weighed in a common balance, and explain your answers.

20. If a mass of a kilogram be placed on a horizontal plane, which is made to descend vertically with an acceleration of 100; find in gravitation measure the pressure on the plane.

21. If a mass of 10 lbs. be placed on a horizontal plane, which is made to ascend vertically with an acceleration of 20 feet per second per second; find in poundals the pressure on the plane.

22. If the velocity of each of the bodies in Attwood's machine be 20 feet per second, when they are at the same height above the ground, and if at that instant the string be cut, find how far apart the bodies will be in 5 second

23. Two bodies of 4 and 5 kilograms together pull on of 6 kilograms over a smooth peg by means of a connecting string; after descending through 10 metres, the 5 kilograms mass is detached without interrupting the motion; find through what distance the remaining 4 kilograms will descend.

Chapter VIII.

Archimedes' Principle.

71. Since the weights of bodies at the same place are directly proportional to their masses, and since different bodies differ in their specific masses or densities, therefore they will also have different *specific weights*, or, as they are more commonly called, specific gravities.

The *specific gravity* of a body is the ratio of its weight to the weight of an equal volume of pure water at 4 °C (its maximum density point) *at the same place.*

The specific gravity of water at 4 °C as well as its density will thus be represented by unity, and it is evident that the numbers which represent the density and specific gravity of the same body are the same. Let M, W, D, S, represent the mass, weight, density, and specific gravity of a body, and m, w, the mass and weight of an equal volume of water at 4 °C, then $M : m = D$, and $W : w = S$, by definition, but (art. 63) $M : m :: W : w$, $\therefore D = S$.

In the case of a body whose specific gravity is not uniform throughout, the above definition gives the *mean specific gravity* of the body.

72. The most convenient methods of determining the specific gravities of liquid and solid bodies depends upon the Principle of Archimedes :

Every body immersed in a fluid is subjected to a vertical upward pressure equal to the weight of the fluid displaced.

The truth of this principle is at once seen when we think that, if the body were replaced with a portion of fluid of

the same kind without any other change, the weight of the fluid would be supported. Its truth is *sensibly* felt in bathing on a shingly beach, when it is found that, the deeper one enters the water, the less are the soles of the feet hurt by the pressure of the stones on them. It can be proved directly by immersing in a liquid, a body, whose volume can be measured exactly (such as a cube, cylinder, or sphere), observing the apparent loss of weight of the body, and then weighing the amount of liquid displaced. In the case of a floating body, the volume of the part of the body immersed is to be calculated, and then the weight of this volume of fluid will be found to be equal to the entire weight of the floating body. Convenient experiments to show these facts are given in books on Experimental Physics.

73. The occasion which led Archimedes to the discovery of this principle was the giving to him by King Hiero of Syracuse the problem :—to discover the amount of alloy which, the king suspected, had been fraudulently put into a crown, which he had ordered to be made of pure gold. It is said that Archimedes saw the solution of the problem one day on entering the bath, and no doubt it was by his observation of the *buoyancy* of the water. It may have been, however, by his noticing that the volume of the water which he displaced would just be equal, by the principle of impenetrability (art. 2), to the volume of the immersed part of his body. Indeed, one of the most important applications of the principle of impenetrability is to determine exactly the volume of any irregularly shaped body, by immersing it in a liquid contained in a measuring glass, and noting the change of level which takes place.

As is pointed out by Prof. Tait in his "Properties of

Matter," it was fortunate for Archimedes that the densities of the alloys of gold are not very different from the means of the densities of the component metals, else the fraudulent goldsmith might have escaped. The discovery of a most important hydrostatic principle was however far more important than the solution of King Hiero's problem. Let us consider how it enables us to determine the specific gravities of liquid and solid bodies.

74. I. *Liquids, to an approximation of the first degree:*

a). By means of a *balance*, either *common* or *spring*.

Weigh a body which is not attacked either by water or the liquid, *e.g.* a piece of quartz, or a platinum ball, firstly in air, secondly in water, thirdly in the liquid whose specific gravity is required:

Let w_1 = weight of the body in air,
w_2 = ... water,
w_3 = ... the liquid,

$$\text{then s. g. of the liquid} = \frac{w_1 - w_3}{w_1 - w_2}$$

b). By means of *hydrometers* or *areometers*.

These are instruments, essentially hollow closed tubes, weighted below, for determining specific gravities by observing how far they sink in water and other liquids, or by observing what weight will make them sink to a certain depth. The latter are called hydrometers of *constant* immersion, the former hydrometers of *variable* immersion.

1. By means of an hydrometer of constant immersion, *e.g.* Nicholson's.

Let w_1 = weight of the hydrometer in air,

w_2 = weight required to sink the hydrometer to the marked depth in water,

w_3 = ditto, ditto, ditto, in the liquid,

$$\text{then s. g. of the liquid} = \frac{w_1 + w_3}{w_1 + w_2}$$

2. By means of hydrometers of variable immersion.

These are called *salimeters* or *alcoholimeters*, according as they are used for liquids of greater or less specific gravity than that of water. Both kinds have scales attached to them which tell either the specific gravity directly for any immersion, or the volume immersed, in which case the specific gravity must be calculated. A thermometer is frequently attached to tell the temperature of the liquid.

c). By means of a *specific gravity bottle.*

Let w_1 = weight of the s. g. bottle empty,

w_2 = ... full of water,

w_3 = ... full of the liquid,

$$\text{then s. g. of the liquid} = \frac{w_3 - w_1}{w_2 - w_1}$$

75. II. *Solids, to an approximation of the first degree :*

a). By means of a balance, either common or spring.

Let w_1 = weight of the body in the air,

w_2 = ... water,

$$\text{then s. g. of the body} = \frac{w_1}{w_1 - w_2}$$

b). By means of an hydrometer of constant immersion.

Let w_1 = weight of the body in air,
w_2 = weight required to make the hydrometer alone sink to the marked depth in water,
w_3 = weight required to make the hydrometer, with the body attached to the lower part of it, sink to the marked depth in water,

$$\text{then s. g. of the body} = \frac{w_1}{w_1 - w_2 + w_3}$$

for evidently if w denote the weight of the hydrometer in air, then $w + w_2$ will be the weight of water displaced by the hydrometer, and $w + w_1 + w_3$ the weight of water displaced by the hydrometer and body together; therefore the difference $w_1 + w_3 - w_2$ will be the weight of water displaced by the body alone: w_1 can easily be determined by the hydrometer, although it is simpler to measure it by means of a common balance.

c). By means of a specific gravity bottle.

This method is particularly convenient for finding the specific gravities of powders.

Let w_1 = weight of the powder.
w_2 = weight of the specific gravity bottle, full of water,
w_3 = weight of the specific gravity bottle, after the powder has been inserted, and the bottle thereafter filled up with water,

$$\text{then s. g. of the powder} = \frac{w_1}{w_1 + w_2 - w_3}$$

d). When a solid body is soluble in water, we find its specific weight relatively to a liquid in which it is insoluble, and multiplying this by the specific gravity of the liquid, we get the specific gravity of the body. As an example let us take common salt, and adopt method *a*).

Let w_1 = weight of the salt in air,
w_2 = weight in kerosene or turpentine of a vessel to hold the salt,
w_3 = weight in kerosene or turpentine of the vessel containing the salt,
s = the s. g. of kerosene or turpentine,

$$\text{then s. g. of the salt} = \frac{w_1 s}{w_1 + w_2 - w_3}$$

Examination VIII.

1. Define the specific gravity of a body.

2. Prove that the numbers which measure the density and specific gravity of any body are the same.

3. Give the specific gravities of a few well-known substances.

4. Enunciate and prove Archimedes' principle.

5. Describe several illustrative experiments which prove the same principle.

6. Explain why a boat built entirely of *iron* can float in water. What is the s. g. of iron ?

7. Give the history of the discovery by Archimedes of his Principle.

8. Is the density of an alloy always equal to the mean of the densities of the component metals ? Give examples.

9. What are the three practical methods of determining the specific gravities of solid and liquid bodies ?

10. Give formulae for all the methods in the case of both solid and liquid bodies.

11. What is an hydrometer? Give the names of the different kinds.

12. When a solid body is soluble in water, how is its specific gravity found?

13. How would you find the specific gravity of (1) sulphuric acid, (2) nitric acid, (3) bichromate of potash, (4) sand, (5) a piece of cork?

14. Given a common balance with a hook to weigh bodies in water, a piece of cork, and a piece of lead sufficient to sink the cork in water; shew (giving formulae) how to determine the specific gravity of the cork.

15. How can the volume of an irregularly shaped stone be accurately determined?

Exercise VIII.

1. A piece of limestone balances 20·21 grams in air, and 12·82 in water; find its specific gravity.

2. An egg whose volume is 48 and s. g. 1·015 is put into salt water whose s. g. is 1·026; find what volume of the egg will be above the surface of the water.

3. An empty bottle hermetically sealed, whose mass is 33, is attached to a heavy body of mass 203, and the whole is found to balance 10 grams in water. The heavy body alone balances 178 grams in water; find the s. g. of the heavy body and of the empty bottle.

4. When 1 lb. of cork is attached to 21 lbs. of silver, the whole is found to balance 16 lbs. in water. If the s. g. of silver be $10\frac{1}{2}$; find that of cork.

5. A person whose mass is 140 lbs. enters the sea to bathe. If the s. g. of sea-water be 1·028, and of the human body 0·9, find the pressure on his feet when three-fourths of his body is immersed.

6. A ball of platinum whose mass is a kilogram, when in water, balances 955 grams; what will it balance when in mercury (s. g. 13·6)?

7. A piece of iron, whose s. g. is 7·5, floats in mercury whose s. g. is 13·6; find what part of the iron is above the surface of the mercury.

8. The s. g. of ice is 0·92, and of sea-water 1·028; find what fraction of an ice-berg is below the surface of the sea.

9. A vessel, containing water, balances 2034 grams; a body of mass 1000, whose s. g. is 8·4, is held in the water; find what will now be the apparent weight of the vessel and water.

10. A piece of cork, whose s. g. is $\frac{1}{4}$, has mass 534; find the pressure necessary to keep the cork under sea-water whose s. g. is 1·028.

11. A kilogram of lead, whose s. g. is 11·35, is suspended in water by a string; find the tension of the string.

12. A vessel partially filled with mercury (s. g. 13·6) balances $72\frac{1}{2}$ kilograms; a kilogram of iron (s. g. 7·5) is held completely immersed in the mercury; what will now be the apparent weight of the vessel and mercury?

13. A block of pine, the volume of which is 4 litres, 340 cub. cm., floats in water with a volume of 2 litres, 240 cub. cm. above the surface; find the s. g. of the pine.

14. Find what force would be necessary to immerse a kilogram of oak (s. g. 0·97) in mercury (s. g. 13·6).

15. A body of mass 58 grams floats in water with two-thirds of its bulk submerged; find its volume.

16. A certain body A is observed to float in water with half its volume submerged, and when attached to another body B of twice its own volume, the combined mass is just submerged; find the specific gravities of A and B.

17. A piece of platinum of mass 15 lbs. is attached to a piece of iron of mass 10 lbs., and the whole is found to balance 1 lb. in mercury (s. g. 13·6); the platinum by itself balances 6 lbs. in mercury; find (1) the s. g. of the platinum, (2) the s. g. of the iron, (3) the weight in water of the iron.

18. Eight cub. ft. of maple (s. g. 0·75) floats in sea water (s. g. 1·026); find what volume of water it displaces.

19. Two bodies, which balance 50 and 60 grams in air, balance each 30 grams in alcohol (s. g. 0·8); compare their volumes and densities, and find the s. g. of the first.

20. A man whose mass is 63·5 kilograms can just float in fresh water; find the maximum weight he could bear up clear of the water, when floating in the sea (s. g. 1·028).

21. How much lead (s. g. 11·35) will a kilogram of cork (s. g. 0·25) keep from sinking in salt-water (s. g. 1·028)?

22. A piece of hard wood of mass 7·6 grams is attached to the lower part of Nicholson's hydrometer, and it is then found that the buoyancy of the hydrometer in salt water is just the same as before the wood was attached, viz., 12·6 grams-weight. If 1·03 be the s. g. of the salt water, find that of the wood.

Chapter IX.

Weight of Gases.

76. We are aware from the effects of wind in driving windmills, ships, &c., that air has mass. That it has weight like solid and liquid bodies can easily be proved by the following experiments :

Exp. 1. Weigh a globe with a tightly fitting stop-cock, firstly full of air, secondly after the air has been extracted from it by an air-pump.

Exp. 2. Boil water in a flask until all the air is driven out, cork it up tightly, weigh when cool, admit air and weigh again.

Exp. 3. Instead of extracting air from the globe in exp. 1, compress air into it, when it will be found to become heavier.

Exp. 4. Weigh the globe in exp. 1 when filled, firstly with air, secondly with hydrogen, thirdly with carbonic acid.

The second and third experiments are due to Galileo; from the third the specific gravity of air may be roughly measured by collecting the compressed air in a pneumatic trough. The fourth experiment proves that gases, like liquid and solid bodies, differ in specific gravity.

77. The fact that gases have weight, and even flame, which is essentially incandescent gas, was known to the Epicureans, if we take Lucretius as their mouth piece. In his great poem, "De rerum natura," written about 56 B.C., he says :

See with what force yon river's crystal stream
Resists the weight of many a massy beam :
To sink the wood, the more we vainly toil,
The higher it rebounds with swift recoil :
Yet that the beam would of itself ascend,
No man will rashly venture to contend :
Thus too the flame has weight, though highly rare,
Nor mounts but when compelled by heavier air.

78. Before considering how the specific weights of gases have been determined, it will be necessary to know how the density of a gas depends upon its pressure or tension. The physical law which tells us this is known as Boyle's law, which may be enunciated in either of the following ways :

In a gas, far removed from its point of condensation, the volume at a given temperature is inversely proportional to the pressure. $PV = C$, where C is a constant so long as we keep to the same gaseous body. Or thus :

In a gas, far removed from its point of condensation, the density at a given temperature is directly proportional to the pressure. $D \propto P$.

The truth of Boyle's law depends of course entirely upon experimental evidence. Dry air is an example of a gas far removed from its point of condensation, and for dry air at all ordinary pressures and temperatures the law may be said to be exact. In the case of solid and liquid bodies it is not necessary to consider the pressure to which they are subjected, for the small changes of density, arising from the changes of pressure to which bodies are in general exposed, would be immeasurable. The case of gases is very different.

79. It was the great French physicist, Regnault, who

first overcame the experimental difficulties necessary to an exact determination of the specific weights of gases. The secret of his success depended upon counterbalancing the globe containing the gas he was weighing, with another globe of equal volume and weight as nearly as possible, so that it was unnecessary for him to make any corrections for barometric, thermometric, or hygrometric changes in the state of the atmosphere during the time of experimentation. Having several times exhausted one of these globes and filled it with a dried gas until he was satisfied that the globe was thoroughly dry, he put the globe in a mixture of ice and water (0° C), and filled it once again with the dried gas at the pressure of the atmosphere, say P cm. of mercury at 0° C. He then partially exhausted the globe, to pressure p say, keeping it at the same temperature 0° C, and noted the change of weight. This change of weight (denote it by w) will, by Boyle's law, be the weight of the gas at 0° which would fill the globe at pressure $P-p$; therefore the weight of gas at 0° required to fill the globe at the mean atmospheric pressure will, by the same law, be $w \times 76 \div (P-p)$.

In this way Regnault found the weights of equal volumes of dry air and other gases. It remained for him to determine the specific weight of dry air with respect to the standard substance, pure water at 4° C. If w' denote the difference of weight between the globe when filled with water at 0°, and when filled with dry air at 0° and pressure P, then $w' + w \times P \div (P-p)$ will denote the weight of the water which the globe would hold at 0°. If therefore s denote the s. g. of pure water at 0°, the s. g. of dry air at 0° and under the mean atmospheric pressure will be

$$\frac{76\, w\, s}{P-p} \div \left(w' + w\frac{P}{P-p}\right)$$

If it be necessary to take into consideration the buoyancy of the air in determining w and w', the methods indicated in the next chapter will explain how this can be done. The following table gives the results of some of Regnault's experiments:

	Mass of 1 litre at 0° and 76 cm.	Specific gravity.
Air (dry)	1·293187	0·0012932
Oxygen	1·429802	0·0014298
Nitrogen	1·256167	0·0012562
Carbonic Acid	1·977414	0·0019774
Hydrogen	0·089578	0·0000896

80. On account of the small densities of gases it is generally more convenient to measure their specific weights with respect to dry air or hydrogen as a standard, than with respect to water at 4 °C.

The specific weight of a gas with respect to dry air (or hydrogen), is defined as the ratio of the weight of any volume of the gas at 0 °C and under the mean atmospheric pressure, to the weight at the same place of an equal volume of dry air (or hydrogen) at the same temperature and pressure.

The following table gives the specific weights of the above gases with respect to dry air and hydrogen:

Hydrogen	0·0693	1·000
Nitrogen	0·9714	14·023
Air (dry)	1·0000	14·436
Oxygen	1·1056	15·962
Carbonic Acid	1·5291	22·075

81. The numbers in the last column and similar results have enabled chemists to establish a most important law

relating to the molecular volumes of gaseous bodies. It may be expressed thus :

The specific gravities of gases, far removed from their points of condensation, are, at the same pressure and temperature, directly proportional to their molecular weights.

In the following table for gases the specific gravities have been calculated from the molecular weights, with the exception of those of nitrogen, air, and oxygen, which are Regnault's experimental measurements.

By means of a good air-pump hydrogen can be rarified to a density 10^4 times less than what it has under the mean atmospheric pressure. We thus see, by comparing the density of platinum with that of rarified hydrogen, what a very great range of density there is, even in substances which can be easily obtained. It will be seen from the following table that the density of a solid substance depends to a certain extent on the way in which it has been prepared. Different specimens of the same material may be found to have different densities arising from the presence of impurities. Even in the case of natural bodies like sapphires or diamonds, different specimens from different places are found to vary in density. In mixtures the density is not always the mean of the densities of the component parts. Thus bell metal has a greater density than the mean of the densities of the component metals ; so with a mixture of alcohol and water. In the case of woods different parts of the same tree may vary in density, as well as specimens from different trees of the same species. Liquids can be obtained more easily in a state of purity, but in such liquids as blood, milk, or sea-water, slight differences of density may be found in different specimens.

Table of Densities and Specific Gravities.

I. Solids at 0° *C.*

Platinum, stamped	22·10	Sapphire	4·01
" rolled	22·07	Diamond	3·52
" cast	20·86	Glass	2·5 to 3·3
Gold, coin	19·36	Kingston Limestone	2·70
" cast	19·26	Rock-crystal (Quartz)	2·66
Lead, cast	11·35	Ivory	1·92
Silver, cast	10·47	Anthracite	1·80
Copper, hammered	8·88	Ebony, American	1·33
" cast	8·79	Mahogany, Spanish	1·06
Bronze, Brass, } average	8·40	Box, French	1·03
		Oak, English	0·97
Steel	7·82	Apple	0·79
Iron, wrought	7·79	Maple	0·75
" cast	7·21	Walnut	0·68
Tin, cast	7·29	Elm	0·70
Zinc, cast	7·00	Willow	0·58
Aluminium	2·67	Poplar	0·38
Magnesium	1·74	Cork	0·24

II. Liquids at 0° *C.*

Mercury	13·596	Water at 4°	1·000000
Sulphuric Acid	1·84	" at 0°	0·999873
Human Blood	1.05	Olive Oil	0·915
Milk, of cow	1·03	Alcohol	0·80
Sea-water	1·027	Sulphuric Ether	0·72

III. Gases at 0° *C and* 76 *cm. pressure.*

Hydrogen	0·0000896	Air (dry)	0·0012932
Ammonia	0·0007619	Oxygen	0·0014298
Aqueous Vapour	0·0008044	Sulphuretted Hydrogen	0·0015219
Carbonic Oxide	0·0012510	Carbonic Acid	0·0019658
Nitrogen	0·0012562	Chlorine	0·0031684

Examination IX.

1. How do we become aware that air possesses the property of mass ?

2. Describe three experiments which prove that air has weight.

3. How is it proved that gases differ like solid and liquid bodies in specific gravity ?

4. Enunciate Boyle's law in two ways, and shew that the one follows from the other.

5. What was the secret of Regnault's success in weighing gases ?

6. Describe fully Regnault's method of determining the specific gravity of dry air.

7. Define the specific weight of a gas with respect to dry air, and also with respect to hydrogen.

8. What is the specific gravity of dry air, 1) with respect to water at 4°, 2) with respect to hydrogen ?

9. Enunciate the law which connects the specific gravity of a gas with its molecular weight.

10. What is the range of density as found by experiment ?

Exercise IX.

1. Determine the mass and weight of 10 litres of oxygen at 0° and at a pressure of 74 cm. of Hg. at 0° in the latitude of Kingston, Ont.

2. Determine the pressure in prems under which chlorine has a density 3 with respect to air.

Chapter X.

Exact Specific Weights.

82. Since the principle of Archimedes evidently applies to gases as well as to liquids, all bodies in the atmosphere are subjected to a vertically upward pressure equal to the weight of the air displaced by them. This may be illustrated experimentally by the *baroscope* and *balloons*. In determining the specific weights of solid and liquid bodies to an approximation of the first degree, we neglected the buoyancy of the surrounding atmosphere. Let us now determine these specific weights to an approximation of the second degree. This is done by taking into consideration the buoyancy of the air, but reckoning its specific gravity as $1 \cdot 2932 \times 10^{-3}$, without noting what may be its barometric, thermometric, and hygrometric states. The following problem will illustrate the process:

Given w_1, w_2, w_3, the number of grams which balance a solid body in air, water, and another liquid respectively; to determine the specific gravities of the solid body and liquid to an approximation of the second degree.

Let R denote 0·0012932 gram-weight. The approximate volume of the solid body will be $(w_1 - w_2)$ cub. cm., and therefore the weight of air displaced by the solid body will be $(w_1 - w_2)\ R$ gram-weight approximately.

Let s denote the s. g. of the standard masses against which the body is weighed. This should be determined by the maker of the standard masses. Then $w_1 \div s$ is the volume of the standard masses which balance the body in air, and therefore $(w_1 \div s)\ R$ the approximate weight of

air, in grams-weight, displaced by them. Therefore the approximate weight of the solid body in vacuo

$$= w_1 - \frac{w_1}{s} R + (w_1 - w_2) R \text{ grams-weight} = W_1$$

The approximate weight of the solid body in water

$$= w_2 - \frac{w_2}{s} R \text{ grams-weight} = W_2$$

The approximate weight of the solid body in the liquid

$$= w_3 - \frac{w_3}{s} R \text{ grams-weight} = W_3$$

Then the specific gravity of the solid body $= \dfrac{W_1}{W_1 - W_2}$

and .. .. liquid body $= \dfrac{W_1 - W_3}{W_1 - W_2}$

each to an approximation of the second degree.

83. To get the specific weight of a body to a degree of approximation of the third degree, we require to calculate the density of air at the pressure, temperature, and hygrometric state in which it is at the time the weighings are performed, as well as to allow for the temperature of the water in which the body is weighed. We have already learned from Boyle's law (art. 78) how the density of a gas depends upon its pressure. The law of change of density of a gas, arising from change of temperature, was first discovered by Charles. It may be enunciated thus :

The dilatation of a gas, far removed from its point of condensation, and at a constant pressure, is directly proportional to the increase of temperature; and the coefficient of dilatation is the same for all gases.

If the dilatation be reckoned from $0° C$, the coefficient of dilatation is very approximately 0·003665 or $\frac{1}{273}$. Hence

if V_t be the volume at temperature $t°$, and V_0 the volume at $0°$, we may express the law algebraically thus:

$$V_t = V_0\,(1+0{\cdot}003665\,t), \text{ or } V_t = V_0\left(1+\frac{t}{273}\right)$$

If now temperature be reckoned from the zero of the air thermometer, *i.e.* from $-273°$ C, Charles' law may be expressed thus:

The volume of a gas, far removed from its point of condensation, is, at a constant pressure, directly proportional to its temperature reckoned from the zero of the air thermometer. Or thus:

The density of a gas, far removed from its point of condensation, is, at a constant pressure, inversely proportional to its temperature reckoned from the zero of the air thermometer. Or thus:

The pressure of a gas, far removed from its point of condensation, is, at a constant volume, directly proportional to its temperature reckoned from the zero of the air thermometer.

The equation $PV = CT$ evidently expresses both Boyle's and Charles' laws. The quantity C is constant, so long as we keep to the same gaseous body, however the pressure, volume, or temperature be changed: *T must be reckoned from the zero of the air thermometer.*

84. To allow for the hygrometric state of the air, we require first to know the law of Dalton relating to the pressure of a mixture of gases. It may be enunciated thus:

When two or more gases, which do not act chemically on one another, are enclosed in a vessel, the resultant pressure is the sum of the pressures of the gases when placed singly in the vessel.

The physical principle underlying Boyle's and Dalton's laws has been beautifully expressed by Rankine thus: *When one, or more gases, which do not act chemically on one another, is confined in a vessel, each portion of gas, however small, exerts its pressure quite independently of the presence of the rest of the gas in the vessel.* Dalton's law tells us that the pressure of moist air is just the sum of the pressures of the dry air and aqueous vapour mixed with it.

85. From the classical experiments of Regnault the pressure of the aqueous vapour in the atmosphere can be determined, so soon as the *dew-point* is known. The dew-point is the temperature at which the atmosphere at any place would be saturated with the aqueous vapour which it contains. It is found experimentally by means of an *hygrometer*.

The following is a part of Regnault's table of the maximum pressures (or pressures of saturation) of aqueous vapour at different temperatures. It gives the pressure of the aqueous vapour in the atmosphere, in centimetres of mercury at 0° at the latitude of Paris, for dew-points from 0° to 20° C.

Temp.	Pressure.	Temp.	Pressure.	Temp.	Pressure.	Temp.	Pressure.
0°	0·4600	5°	0·6534	10°	0·9165	15°	1·2699
1°	0·4940	6°	0·6998	11°	0·9792	16°	1·3536
2°	0·5302	7°	0·7492	12°	1·0457	17°	1·4421
3°	0·5687	8°	0·8017	13°	1·1162	18°	1·5357
4°	0·6097	9°	0·8574	14°	1·1908	19°	1·6346
5°	0·6534	10°	0·9165	15°	1·2699	20°	1·7391

Observe, that whilst the pressure of a gas *far removed from its point of condensation* depends upon its *temperature and volume*, the pressure of the same gas *at its point of condensation* (or, *in contact with its own liquid*) depends

upon its *temperature alone.* The following example will illustrate how the density of the atmospheric air can be calculated when its barometric, thermometric and hygrometric states are known.

Ex. The reading of the barometer is 76·4, the temperature 20°, the dew-point 8°, and the latitude 44° 13′; to determine the density of the air, given the coefficient of dilatation of the barometer scale to be 0·000018, and the mean coefficient of dilatation of mercury between 0° and 20° to be 0·00017951.

The barometric pressure in centimetres of mercury at 0° in the latitude of Paris will be

$$= \frac{76{\cdot}4\,(1+20\times 0{\cdot}000018)}{1+20\times 0{\cdot}00017951} \times \frac{980{\cdot}54}{980{\cdot}94} = 76{\cdot}1230.$$

This pressure is due, according to Dalton's law, partly to dry air, and partly to the aqueous vapour in the air. According to Regnault's tables the pressure of the aqueous vapour for the dew-point 8° is 0·8017. Therefore the pressure of the dry air in the atmosphere is 75·3213. Hence, applying Boyle's and Charles' laws, the density of the dry air in the atmosphere

$$= 1{\cdot}2932\times 10^{-3} \times \frac{75{\cdot}3213}{76} \times \frac{273}{293} = 1{\cdot}1942 \times 10^{-3},$$

the density of the aqueous vapour in the atmosphere

$$= 8{\cdot}044\times 10^{-4} \times \frac{0{\cdot}8017}{76} \times \frac{273}{293} = 7{\cdot}9\times 10^{-6},$$

$\therefore$ the density of the atmospheric air $= 1{\cdot}2021\times 10^{-3}$.

86. *To determine the specific gravity of a solid or liquid body to an approximation of the third degree.*

Ex. Given w_1, w_2, w_3, the number of grams which balance a solid body in air, distilled water, and another liquid

respectively; the reading of the barometer 76·4, the temperature 20°, the dew-point 8°, and the latitude 44° 13′; the coefficient of dilatation of the barometer scale 0·000018, the mean coefficient of dilatation of mercury between 0° and 20°, according to Regnault, 0·00017951, and the density of distilled water at 20°, according to Despretz, 0·998213; to determine the specific gravities of the solid body and liquid to an approximation of the third degree.

Find, as in art. 82, W_1 the approximate weight of the solid body in vacuo, S the s. g. of the solid body to an approximation of the second degree, and, as in last article, R the density of the atmospheric air. Denote by s the s. g. of the standard masses against which the body is weighed, and by $\overline{S}$ the s. g. of distilled water at 20°.

The weight of the solid body in vacuo (in grams-weight) *very* nearly $= w_1 + (\frac{W_1}{S} - \frac{w_1}{s}) R = \overline{W_1}$

The weight of the body in distilled water at 20°

$$= w_2 - \frac{w_2}{s} R = \overline{W_2}$$

The weight of the body in the liquid at 20°

$$= w_3 - \frac{w_3}{s} R = \overline{W_3}$$

Then s. g. of the solid body at 20° $= \frac{\overline{W_1}}{\overline{W_1} - \overline{W_2}} \overline{S}$

and s. g. of the liquid at 20° $= \frac{\overline{W_1} - \overline{W_3}}{\overline{W_1} - \overline{W_2}} \overline{S}$

to an approximation of the third degree.

If the coefficients of dilatation of the solid body and liquid be known, the specific gravities at any other temperature may be determined. By taking the specific gravity of the solid body just determined in place of S, and $\overline{W_1}$ in

in place of W_1, and repeating the method above, we could find the specific gravities to an approximation of the fourth degree, and so on to higher degrees. This would however be useless, as the errors of experimentation would certainly be greater than any errors, from the exact values, of the specific gravities to an approximation of the third degree.

Examination X.

1. How can it be proved experimentally that Archimedes' principle applies to gases?

2. Given the apparent weights of a solid body in air, water, and another liquid, to determine the specific gravities of the solid body and liquid to an approximation of the second degree.

3. Enunciate in four ways the law of Charles, and deduce each one from the others.

4. Enunciate Dalton's law relating to the pressure of a mixture of gases. Give Rankine's statement of the physical principle underlying Boyle's and Dalton's laws.

5. What are the various corrections to be made in determining the specific gravity of a body to an approximation of the third degree? What are the physical instruments used for this purpose?

6. Define the dew-point. What does it tell us?

7. Write down an algebraical equation which expresses the facts contained in Boyle's and Charles' laws.

Exercise X.

1. The reading of the barometer in a room is 77·34, the thermometer 15°, the dew-point 10°, the latitude 44° 13′;

the coefficient of dilatation of the barometer scale is 0·000018, and the mean coefficient of dilatation of mercury between 0° and 15°, according to Regnault, 0·0001794; the room is 12·5 m. long, 5·45 m. broad, and 3·7 m. high; find the volume, mass, and weight of the air in the room.

2. A piece of cork balances 50 grams in air; when attached to the bottom of Nicholson's hydrometer, it is found that 175 grams-weight are required to sink the hydrometer to the marked depth in distilled water, whilst only 25 grams-weight are required to sink the hydrometer alone; the reading of the barometer is 78 cm., of the thermometer 14°, the dew-point 12°; the latitude, that of Edinburgh; the specific gravity of the standard masses is 8·4, the coefficient of linear dilatation of the barometer scale 0·000018, the mean coefficient of dilatation of mercury between 0° and 14°, according to Regnault, 0·00017937, and the density of distilled water at 14°, according to Despretz, 0·999285; to determine the specific gravity of cork to approximations of the first, second and third degrees.

3. A lump of gold balances 437·008 grams in air, 414·357 in distilled water, and 420·699 in sulphuric ether; the reading of the barometer is 77·3 cm., of the thermometer 9°, the dew-point 4°; the latitude, that of Greenwich; the s. g. of the standard masses against which the body is weighed is 8·4, the coefficient of linear dilatation of the barometer scale 0·000018, the mean coefficient of dilatation of mercury between 0° and 9°, according to Regnault, 0·0001792, and the density of distilled water at 9°, according to Despretz, 0·999812; to determine the specific gravities of gold and sulphuric ether to approximations of the first, second, and third degrees.

Chapter XI.

Energy. *Work.*

87. *Energy* is the power to overcome resistance through space. *Work* is the expenditure of energy, or is the transference of energy from one body to another. Work is physically manifested either by accelerating the motions of material bodies, or by changing the configuration of a material system against resistance. Thus when a man raises a body vertically upwards he does work against the body's weight, and by the work done produces a change of configuration of the material system consisting of the body and the earth. Again, when he throws a cricket ball, he does work in giving the ball motion.

We have firstly defined energy, then work. The order might have been reversed, thus: *work* is the production of motion against resistance; *energy*, the power of doing work.

88. The property of *mass* is essential to any body possessing energy. If further the body be in motion, it has energy in virtue of this motion. Thus, the energy of a moving cannon ball is due entirely to its mass and velocity, and it is this energy which enables it to tear down a rampart against the resisting molecular forces. Similarly, in virtue of its mass and velocity, a running stream can drive a water-wheel and thus grind our corn. These are examples of *visible kinetic energy*.

89. A body may further possess energy in virtue of its position with respect to other bodies, which, along with it, form a material system. There are forces which are constantly acting between every pair of particles of a material system. The force of gravitation, the molecular forces

(cohesion, elasticity, crystalline force, &c.), the atomic force or chemical affinity, are different aspects of force which is found by experience to act between every pair of particles in the universe. When work is done against such force upon a body which forms a part of a material system, so as to alter the configuration of that system, the body in virtue of its new position has energy which it did not previously possess. Thus a *head* of water has energy in virtue of its position with respect to the earth. The *wound up* spring of a clock can keep it going for a week or longer. *Compressed* air, such as is used for the conveyance of letters in Paris and elsewhere, is a store of energy in virtue of the configuration of the aerial particles. These are examples of *visible potential energy*.

A material system, such *e.g.* as the solar system, possesses visible energy; firstly, on account of the motions of its component parts; this is its kinetic energy; secondly, on account of its configuration, *i.e.* the relative positions of its component parts; this is its potential energy. An oscillating pendulum, or a vibrating spiral spring, is a beautiful and simple example of a body whose visible energy is constantly passing from the one form into the other. At the extremities of the line of vibration, the energy is wholly potential; at the middle point, it is wholly kinetic; and at intermediate positions, it is partly kinetic and partly potential. In an undershot water-wheel the miller depends upon the kinetie energy of the water to grind his corn; in an overshot water-wheel he depends upon the potential energy of the water to drive the wheel.

90. Work is measured by the force overcome and the distance through which it is overcome conjointly. Thus in measuring the work done in raising bricks to the top of a

house, the builder multiplies the weight of the bricks by the vertical height through which they are raised. To raise double the number of bricks through double the height will evidently require four times as much work. The *unit of work* is that in which unit of force is overcome through unit of distance. In the C. G. S. system the unit of work is the work of overcoming a dyne through a centimetre, and is called an *erg*. The equation $E = fs$ evidently gives the relation between the work done in ergs, the force overcome in dynes, and the distance in centimetres through which the force is overcome.

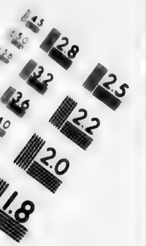
91. *To determine the kinetic energy of a body whose mass is m and velocity v.*

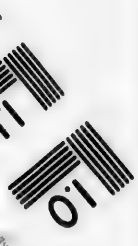
Let the body move against a uniform resistance f. This will give the body an acceleration $f \div m$, opposite in direction to the body's motion. If s be the whole distance through which the body can act against this resistance, so that after passing through the distance s the velocity is zero, by equation (6) art. 35,

$$0 = v^2 - 2\frac{f}{m}s, \quad \therefore fs = \tfrac{1}{2}mv^2$$

but fs is evidently the total work done by the body against the resistance f, and is all that it can do, since, after doing this work, the velocity is zero. Hence $\frac{1}{2}mv^2$ measures the body's kinetic energy, or amount of work the body can do, in virtue of having mass m and velocity v. If m be measured in grams, and v in tachs, $\frac{1}{2}mv^2$ is the measure of the body's kinetic energy in ergs. Since the kinetic energy of a body varies as the *square* of its velocity, it is evident that it has no direction; in this respect it is well to note, energy differs from momentum and force.

92. As an illustration of the preceding article let us





consider the case of a body of weight w and mass m, thrown vertically upwards in vacuo with velocity u. In virtue of its kinetic energy it raises itself against its own weight. If h be the greatest height reached, the work done is wh. Now $w=mg$ (art. 63), and $h=u^2 \div 2g$ (art. 36), therefore $wh=\frac{1}{2}mu^2$, which, as might be expected, is the same result as we got in last article.

Is the energy of the body in its elevated position destroyed? No, it is merely in a latent form; for, without imparting any more energy to the body, we can get out of it, *in virtue of its new position*, the same amount of work as it was capable of doing at starting. This will be at once understood when we remember, that by letting the body fall to its point of starting, it acquires the same velocity which it had at starting (art. 36), and has therefore again the original kinetic energy imparted to it. In its elevated position the energy of the body is called *potential*. Such is the energy of a head of water used to drive machinery, or of the elevated massive bodies whose energy is used to drive piles into the ground.

We see from the above that the potential energy of an elevated body is measured by wh, where w is the body's weight, and h its height above the ground. If w be measured in dynes and h in centimetres, then wh measures the potential energy in ergs.

Cor. From the above and art. 35, equations (3) and (6), it is easily seen, that in any intermediate position of the body between the ground and height h, the energy of the body is partly kinetic and partly potential, and that the total energy is constant and equal to wh or $\frac{1}{2}mu^2$.

93. Just as it is convenient in many practical questions to have a gravitation as well as an absolute unit of force,

so in the practical estimate of work it is often convenient to use a gravitation unit. Such a unit is the *kilogrammetre*, or work done in raising a body of 1 kilogram mass vertically upwards against its weight through the height of 1 metre. Evidently 1 kilogrammetre $= 1000\, g \times 100$ ergs, ($=98054000$ ergs in the latitude of Kingston, Ont).

94. The F. P. S. unit of work is that required to overcome a poundal through the distance of a foot, and may be called a *foot-poundal*. English engineers use as a gravitation unit of work a *foot-pound*, *i.e.* the work done in raising a body of 1 pound mass vertically upwards through the distance of 1 foot. The foot-pound is evidently equal to g or nearly $32\frac{1}{8}$ foot-poundals..

If m be the mass of a body in pounds and v its velocity in feet per second, then $\frac{1}{2}mv^2$ measures its kinetic energy in foot-poundals (art. 91), and therefore $\frac{1}{2}\, mv^2 \div g$, its energy in foot-pounds. Similarly, if a body, whose mass is m pounds and weight w poundals, be at a height of h feet above the earth's surface, it has potential energy measured by wh foot-poundals, *i.e.* mgh foot-poundals, or mh foot-pounds.

95. The *unit rate of working* in the C. G. S. system is 1 erg per second. If H denote the rate of working in ergs per second, f the resistance in dynes, and v the velocity in tachs of the body moved against this resistance, then $H = fv$. This formula suggests the name *dyntach* for the unit rate of working. Watt's *horse-power* is a convenient gravitation unit adopted by English engineers, and is equal to 33000 foot-pounds per minute, or nearly $7{\cdot}46 \times 10^9$ dyntachs. The French *force-de-cheval* is a similar gravitation unit equal to 75 kilogrammetres per second, or nearly $7{\cdot}36 \times 10^9$ dyntachs. Each of these was supposed to re-

present the rate at which a good horse works, but is now allowed to be too high.

96. The examples of energy we have hitherto taken as illustrations are energies of systems, the motions and configurations of whose parts are plainly visible to us. Our grandest sources of energy are, however, derived from systems, the motions and configurations of whose parts are invisible to us. Whence the energy which enables the labourer to dig the ground, the student to pursue his studies, or the horse to draw his load? These are examples of *vital energy* which the man and horse derive from the food they eat and drink, and the air they breathe. The energy of gunpowder, of steam, and of a voltaic battery are other examples of what is called *molecular energy*.

Food and fuel are our principal immediate sources of energy. Thus coal and the oxygen of the air form a system which, before combustion, in virtue of the separation of the atoms of coal and the atoms of oxygen, possesses *potential energy of atomic separation*. During combustion the energy becomes kinetic, and may be communicated to the water in a boiler so as to heat the water and form steam, and through this be used to drive an engine, and by means of the engine do all sorts of mechanical work. Similarly, food and air form a great store of potential molecular energy, which is transformed during digestion into the vital energy by means of which we do our daily work.

Heat, light, and electricity, in their physical aspects, are well defined as forms of molecular energy. Sound forms a sort of connecting link between visible and invisible or molecular energy. The *Transformation of Energy* is the enunciation of the fact:

Any one form of energy may be transformed, directly or indirectly, into an exact equivalent of any other form.

It is principally *plainly visible energy* that is studied in the science of dynamics.

97. Amongst the most important of the recent advances in Physical Science is the measurement of the different forms of molecular energy in dynamical units. Thus the energy of a *unit of heat*, (the heat required to raise the temperature of 1 gram of pure water from 4 °C to 5 °C), is determined experimentally to be equal to 42 million ergs nearly.

From such measurements the very important generalization, known as the *Conservation of Energy*, has been deduced. This principle asserts :

Through whatever forms energy may pass, it cannot be changed in quantity, and hence the total energy in the universe remains constant.

As the *Conservation of Mass* forms the foundation of the science of modern chemistry, the *Conservation of Energy* may be said to form the foundation of the science of modern physics.

Although the total energy in the universe remains constant, it is gradually being transformed into lower forms so as to be less useful to man. This is the principle enunciated by Sir W. Thomson, and known as the *Degradation of Energy :*

The energy of the universe is gradually being transformed into a form in which it cannot be made use of by man, viz., that of uniformly diffused heat.

98. Perhaps the principle force through which energy is

being constantly dissipated, or degraded into the useless form of diffused heat, is friction (art. 60). The direction of this force is always diametrically opposite to the direction of motion, or to that in which motion *would* take place under the influence of the other acting forces. When the surfaces between which friction is called into play are plane, the laws of friction, as determined by experiment, are well defined and may be enunciated thus:

1. *The friction per unit of area is directly proportional to the normal pressure per unit of area, so long as the surfaces in contact are similar and of the same materials.*

$$f = \mu r.$$

2. *When there is sliding motion, the friction is independent of the velocity.*

The first of these laws is generally divided into two, which may be enunciated thus:

(1) *For similar surfaces of the same materials, the total friction varies directly as the total normal pressure.* $F = \mu R$.

(2) *The total friction is independent of the areas of the surfaces in contact.*

The *maximum* amount of friction is called into play when motion is *just about* to take place, being then generally greater than when motion actually takes place. The constant μ, which measures the ratio of the friction to the normal pressure, is then called the *coefficient of friction* for the two surfaces in question. Rankine has shown that the value of μ lies between 0·2 and 0·5 for wood on wood, 0·2 and 0·6 for wood on metals, 0·3 and 0·7 for metals on stone, and 0·15 and 0·25 for metals on metals.

On account of the dissipation of energy through friction

and other causes, a machine does not do as much useful work as the equivalent of the energy imparted to it. The ratio of the useful work done to the energy supplied is called the *efficiency* or *modulus* of the machine. The *duty* of a steam-engine is the amount of useful work performed per unit mass of fuel consumed.

Examination XI.

1. Define energy and work. How is work physically manifested? Give examples.

2. What other property of matter is invariably associated with the possession of energy?

3. Define kinetic energy, and give examples of it.

4. Define potential energy, and give examples of it.

5. How is work measured? Give examples.

6. Name and define the unit of energy and work.

7. Determine the kinetic energy of a body whose mass is m and velocity v.

8. Prove that the potential energy of a body whose mass is m, and height above the earth's surface h, is mgh.

9. Prove that when a body is moving vertically, under no other force than its weight, its total energy is independent of its position.

10. Define a kilogrammetre and foot-pound, and determine their values in absolute measure.

11. If a body of m pounds be moving with a velocity of v ft. per sec.; find its kinetic energy in foot-pounds.

12. Name and define the unit rates of working in absolute and gravitation measures, according to both the C. G. S. and F. P. S. systems.

13. Give various examples of molecular energy, both kinetic and potential.

14. What are our principal immediate sources of energy? Explain your answer.

15. Define the unit of heat, and give its measurement in dynamical units.

16. Enunciate the principles known as the Transformation, Conservation, and Degradation of Energy.

17. Enunciate the laws of friction for plane surfaces, and define the coefficient of friction.

18. Define the modulus of a machine, and the duty of a steam-engine.

Exercise XI.

1. How much work must be done to pump 1000 cub. ft. of water from a mine 150 fathoms deep?

2. In pile-driving 30 men raised a rammer of 500 kilograms through a height of 40 metres 12 times in an hour; find the rate of working per man?

3. How many ergs are stored up in a mill-pond near Kingston, Ont., which is 40 m. long, 20 m. broad, and 1 metre deep, and has a fall of 5 metres?

4. A ball of 40 lbs. is moving at the rate of 300 miles per hour; find its kinetic energy in ft.-lbs.

5. A machine (modulus $\frac{2}{3}$) for raising coals is worked by two horses; how much coal will be raised in a day of 8 working hours from a pit 90 metres deep?

6. An engine is found to raise 5 tons of material per hour from a mine 110 fathoms deep; find the H. P. of the engine, supposing $\frac{1}{6}$ of its energy to be lost in unavoidable resistances.

7. A railway train of 60 tons in passing over a certain mile has its velocity increased from 40 to 50 miles per hour. If the average friction be 10 lbs.-wt. per ton, find the work done by the engine in passing over the mile, and the kinetic energy of the train at the end of the mile.

8. What must be the horse-power of an engine whose modulus is $\frac{4}{5}$, working 8 hours per day, which supplies 3000 families with 100 gallons of water each per day, the mean height to which the water is raised being 60 feet?

9. How many bricks will a labourer raise to the mean height of 20 ft., working 8 hours per day; given that the volume of 17 bricks is a cubic ft. and the mass 125 lbs., and that the average rate of doing such work is 1200 ft.-lbs. per minute?

10. If a load be 10 bricks (Ex. 9), and the man's own mass 140 lbs., what is the rate at which he expends his vital energy when working?

11. What would be the cost per ton to raise coals from a pit 25 fathoms deep, allowing \$3 per day for a horse and driver, and that the horse performs 22000 effective units of work per minute, working 8 hours per day?

12. At what rate will a train of 80 tons be drawn by a locomotive-engine of 70 H.P., the frictional resistance being 10 lbs.-wt. per ton; and how far, after steam is shut off, will it go before being brought to rest?

13. If 8 lbs.-wt. per ton (Ex. 12) be the average frictional

resistance until full speed is attained, how long will it take for the train to attain its maximum speed after starting; and how far will it have travelled in this time?

14. In what time will a locomotive of 100 force-de-cheval, drawing a train of 100 tonnes, complete a journey of 100 kilometres, supposing that the frictional resistance until full speed is attained, and after steam is shut off until it stops, be on the average 3 kilograms-weight per tonne, and after full speed is attained, 4 kilograms-weight per tonne. (1 tonne $= 10^6$ grams).

15. According to Navier it requires 43333 kilogram-metres to saw through a square metre of green oak; how many ares of oak planking will an engine of 5 force-de-cheval cut in a day of 8 hours?

16. If 5 H. P. be applied to a machine which lifts 3000 cub. ft. of water per hour to the height of 40 feet, what is the modulus of the machine?

17. The section of a mill stream is 4 ft. by 2 ft., the mean velocity 20 ft. per min., and the height of the fall 15 ft.; what will be the H. P. of the water-wheel whose modulus is 0·7, and how many cub. ft. of water will the wheel pump per day to the height of 100 ft.?

18. The mean annual rainfall over the earth's surface is estimated at about 5 feet, and 1000 feet the mean height from which rain falls; calculate the sun's horse-power used in *merely raising* the water in the process of evaporation. (Earth's mean radius = 20902070 feet).

19. Determine the H. P. of the river Niagara which has a total descent of 334 feet, and discharges about 4×10^7 tons of water per hour.

20. A ball of 10 kilograms is fired from the mouth of a cannon 3 metres long with the velocity of 15 kilotachs; find the mean pressure of the gaseous products on the ball.

21. There were 4000 cub. ft. of water in a mine of depth 60 fathoms, when an engine of 70 H. P. began to work the pump; the engine worked for 5 hours before the mine was cleared of the water; if the modulus of the engine were $\frac{2}{3}$, find the rate at which water was entering the mine.

22. A steam-engine raises 70 cub. ft. of water per min. from a depth of 800 ft.; how many tons of coal are burned per day, supposing the duty of the engine to be $\frac{1}{2}$ million ft.-lbs. per lb. of coal.

23. Find in dyntachs the rate at which a fire-engine works, which discharges 10 kilograms of water per second with a velocity of 1500 tachs.

24. A railway carriage of two tons mass is started on a level railroad with a velocity of 8 ft. per sec., and moves over 400 ft. before it stops; determine the coefficient of frictional resistance.

25. A cistern is 10 ft. long, 7 ft. broad, and 8 ft. deep. The height of the bottom of the cistern from the water in the well is 56 ft. If a man can work with a pump at the rate of 2600 ft.-lbs. per minute, and the modulus of the pump is 0·66, how long will he take to fill the cistern?

26. A spring tide raises the level of the river Thames, between London and Battersea bridges, on an average 15 feet. If 5 miles be the distance between the bridges and 900 feet the mean breadth of the river, find the potential energy of the spring tide when full.

Chapter XII.

Action and Reaction.

99. It has been pointed out, that whenever a force acts, there are always two bodies concerned. We generally speak of one of the two as receiving a change of momentum, and of the other as being concerned in the production of this change. Newton clearly pointed out in his third law of motion that the action was a mutual one; that change of momentum was received by both bodies, equal in magnitude but diametrically opposite in direction:

To every action there is always an equal and contrary reaction; or, the mutual actions of any two bodies are always equal (in magnitude) *and opposite in direction.*

The word *force* is properly used when we consider the effect of the action between any two bodies in changing the momentum of one of them only. *Stress* is a term applied to the mutual action between any two bodies, when there is special reference to the *dual* character of that action, as enunciated by Newton. This third law tells us that all dynamical actions between bodies are of the nature of stresses. When a body falls to the ground under the action of its weight, the earth rises to meet it with an equal momentum. Since, however, the mass of the earth is so very much greater than that of any body on its surface, the motion of the earth is so small that it may be neglected. When two like magnetic poles, free to move, are brought near one another, it will be found that the repulsion is mutual. When the loadstone attracts a piece of iron, the iron attracts the loadstone with an exactly equal force. When the table is pressed by the hand, we feel that the hand is likewise pressed by the table. When a horse draws a

canal boat by means of a stretched rope, the horse is drawn backwards as much as the boat is drawn forwards. This may easily be proved by cutting the rope, when immediately the horse falls forwards. This is further easily seen, when we reflect that relatively to the boat the horse does not move at all. According to Archimedes' principle, any body immersed in a fluid is subjected to a vertically upward pressure equal to the weight of the fluid displaced. The fluid, on the other hand, is subjected to a vertically downward pressure equal to the weight of the fluid displaced. This can be easily shewn experimentally by balancing a vessel filled with water in a common balance, and immersing in the water a body held by a cord. The equilibrium will be immediately destroyed, and the force necessary to restore equilibrium will be found to be equal to the weight of the water displaced. When two railroad trains or other bodies collide, the change of momentum in the one is just equal, and opposite in direction, to the change of momentum in the other, whatever be the original direction or rate of motion of either train.

100. Since momentum has direction as well as magnitude, it at once follows from the above law, that the total momentum of two bodies is not altered by their mutual action. From this the important principle called the *Conservation of Momentum* is at once deduced:

The total momentum of any body, or system of bodies, cannot be altered by the mutual actions of its several parts.

As an illustration of this principle let us consider the *kick* of a gun. Here we have a system consisting of 3 bodies, the gun, the gas formed from the gunpowder, and the ball; it will be at once seen that the backward momentum of the gun is just the equivalent of the forward momentum of the ball.

The total momentum of the universe is a constant quantity is an immediate deduction from the same principle.

101. The *Conservation of Momentum* teaches us that change of momentum in a body or system of bodies must be produced by forces *external* to the body or system. Let any forces act upon a body of mass m and produce in it an acceleration a, then ma is the measure of the single force which would produce the same dynamical effect on the body. If, after Newton, we call a force measured by $-ma$ the *resistance to acceleration*, which the body offers in virtue of its mass and inertia, then *D'Alembert's Principle* at once follows as a corollary to Newton's third law:

The external forces acting upon a body (or system of bodies), together with the resistance (or resistances) to acceleration, form a system of forces in equilibrium.

This principle evidently amounts to saying that the molecular or internal forces acting within a body or system of bodies are themselves a system of forces in equilibrium.

102. Newton published his axioms or laws of motion in his celebrated work "Principia Philosophiae Naturalis". In the scholium appended to his third law he points out that an additional meaning may be attached to the words action and reaction besides that of force. Tait and Thomson in their treatise on Natural Philosophy have recently brought this passage to light and thus translated it:

If the action of an agent be measured by the product of its force into its velocity; and if, similarly, the reaction of the resistance be measured by the velocities of its several parts into their several forces, whether these arise from friction, cohesion, weight, or acceleration; action and reaction, in all combinations of machines, will be equal and opposite.

As pointed out by Tait and Thomson, this remarkable passage contains in it nothing less than the foundation of that great modern generalization, the Conservation of Energy.

103. *Two heavy bodies are connected by an inextensible string which passes over a fixed smooth peg, (or pully, as in Attwood's machine); required to determine the tension of the string.*

Let T denote the tension of the string, m and m' the masses of the bodies, m being the greater. Since the tension of the string is the same throughout by Newton's third law, the acceleration of the heavier body will be $(mg - T) \div m$ downwards, and of the lighter body $(T - m'g) \div m'$ upwards; since these must be equal,

$$g - \frac{T}{m} = \frac{T}{m'} - g\ , \quad \therefore \quad T = \frac{2mm'}{m+m'}g$$

Cor. The acceleration $= \dfrac{mg-T}{m}$ or $\dfrac{T-m'g}{m'} = \dfrac{m-m'}{m+m'}g$

as already proved (art. 70). If $m = m'$, the tension of the string is mg, and there is no acceleration, so that the bodies must either be at rest or moving with uniform velocity.

The above completes the solution of the problem of Attwood's machine (art. 70), when the weight of the string, the pully's mass, and the friction may be neglected.

104. As an additional illustration of Newton's third law let us consider one of the very simplest cases of impulsive force (art. 57), viz., the direct impact of two spheres of uniform density without rotation. If the centres of two spheres move in the straight line joining them, and one impinges on the other, the impact is called direct; otherwise, the impact is called oblique.

Denote by m_1, m_2, the masses of the spheres, and by u_1, u_2, their velocities before impact. If the direction of u_1 be called $+$, u_2 will be $+$ or $-$ according as m_2 is originally moving in the same or opposite direction to m_1. The action which takes place during impact may be explained thus:

a). Alterations of form and volume take place by work being done against the molecular forces, until the relative velocity of the two bodies is destroyed. If R denote the total force called into play during this first stage of the impact, and v denote the common velocity, we get from Newton's second and third laws

$$R=m_1(u_1-v)=m_2(v-u_2) \quad \ldots \quad \ldots \quad \ldots \quad (a)$$

whence $$v=\frac{m_1u_1+m_2u_2}{m_1+m_2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

and $$R=\frac{m_1m_2}{m_1+m_2}(u_1-u_2) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

These equations contain the complete solution of the problem, if the bodies do not separate again after impact. This will be the case, when the force of *adhesion* between the bodies counterbalances the force of *elasticity*, which tends to separate them.

b). If the bodies be sufficiently elastic, they have the common velocity v only for an instant, for an amount of *potential energy of molecular separation* has been stored up in consequence of the change of configuration of the material particles of each sphere, and, in the transformation of this energy into the kinetic form, the molecular forces continue acting, until the original forms and volumes are as much as possible restored. During this second stage of the impact it is evident that the bodies receive accelerations of

momentum in the same directions as during the first stage, and if R' denote the total force called into play,

$$R' = m_1(v - v_1') = m_2(v_2 - v) \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad (b)$$

Now it has been proved by experiment, that if the impact do not make any sensible permanent alteration of form, the relative velocity of the bodies after impact bears a constant ratio to the relative velocity before impact, *i.e.* $v_1 - v_2 = -e(u_1 - u_2)$, where e is a proper fraction, whose value depends only upon the material natures of the spheres. From this and equations (a) and (b) we deduce by algebraical analysis $R' = eR$. Also,

$$v_1 = u_1 - \frac{m_2}{m_1 + m_2}(1+e)(u_1 - u_2) \quad \ldots\ldots \quad \ldots\ldots \quad (3)$$

$$v_2 = u_2 + \frac{m_1}{m_1 + m_2}(1+e)(u_1 - u_2) \quad \ldots\ldots \quad \ldots\ldots \quad (4)$$

$$R + R' = \frac{m_1 m_2(1+e)}{m_1 + m_2}(u_1 - u_2) \quad \ldots \quad \ldots\ldots \quad \ldots \quad (5)$$

The value of e was found by Newton to be $\frac{5}{9}$ for balls of compressed wool and steel, $\frac{8}{9}$ for balls of ivory, and $\frac{15}{16}$ for balls of glass. It is called by most writers the *coefficient of elasticity*, a name strongly objected to by Tait and Thomson, who call it the *coefficient of restitution*. When $e = 1$, the bodies are called perfectly elastic, a condition never perfectly realized; when $e = 0$, the bodies are called inelastic.

Cor. 1. If $m_2 = \infty$, and $u_2 = 0$, the case is that of a sphere impinging normally on a fixed plane. The equations (3), (4), (5), become then

$$v_1 = -eu_1 \ , \ v_2 = 0 \ , \ R + R' = m_1(1+e)u_1 .$$

Cor. 2. If $m_1 = m_2$, and $e = 1$, then $v_1 = u_2$, and $v_2 = u_1$, *i.e.* the bodies interchange velocities. This may be shown

experimentally to be nearly the case for balls of ivory or glass. Also, if $u_2=0$, and $m_1=em_2$, then $v_1=0$, and $v_2=eu_1$.

105. The following results are at once deduced from the preceding investigation:

1. *Whether the bodies be elastic or not, the total momentum is not affected by the impact.*

$$(m_1+m_2)v=m_1u_1+m_2u_2\text{, and } m_1v_1+m_2v_2=m_1u_1+m_2u_2.$$

This is just a particular case of art. 100.

2. *The total visible kinetic energy after impact is less than before impact.*

$$\tfrac{1}{2}(m_1+m_2)v^2=\tfrac{1}{2}m_1u_1{}^2+\tfrac{1}{2}m_2u_2{}^2-\tfrac{1}{2}\,\frac{m_1m_2}{m_1+m_2}(u_1-u_2)^2,$$

and $\tfrac{1}{2}m_1v_1{}^2+\tfrac{1}{2}m_2v_2{}^2=$

$$\tfrac{1}{2}m_1u_1{}^2+\tfrac{1}{2}m_2u_2{}^2-\tfrac{1}{2}\,\frac{m_1m_2}{m_1+m_2}(1-e^2)(u_1-u_2)^2.$$

What becomes of the visible kinetic energy lost? It is transformed into the *molecular kinetic energy of heat*, so that the bodies after impact are warmer than before impact.

Examination XII.

1. Enunciate Newton's third law of motion, and give numerous illustrations of it.

2. What is the difference between the meanings of the terms force and stress?

3. Enunciate and prove the Conservation of Momentum.

4. Explain the kick of a gun after it is fired.

5. Enunciate and explain D'Alembert's principle.

6. How can it be said that Newton in his third law laid the foundation of the science of energy?

7. A string passing over a smooth peg connects two heavy bodies; determine its tension, 1) when the bodies have different weights, 2) when the weights are the same.

8. Two spheres of uniform density without rotation impinge directly; describe the nature of the impact, and determine the equations of motion.

9. What is denoted by e in the theory of impact? How is it experimentally determined? Give its value for certain substances.

10. How do we deduce the equations of impact of a sphere on a fixed plane. Give the equations.

11. Determine under what conditions will two spheres, impinging directly, interchange velocities?

12. Prove that the momentum of a system of uniform spheres without rotation is not altered by direct impacts of its component parts.

13. Determine the loss of visible kinetic energy when two spheres impinge directly. What becomes of it?

Exercise XII.

1. A boulder of 2 tonnes mass is rolled from the summit of El Capitan in the Yosemite valley, a rock rising vertically through the height of 3000 feet; find the velocity of, and distance travelled by, the earth when the boulder strikes the ground, in virtue of their mutual attraction. (See 6, Ex. V).

2. An 81 ton gun is charged with a shot of 100 lbs. mass; if the ball leave the gun with a velocity of 200 ft. per sec.; find the velocity of recoil of the gun.

3. Find the tensions of the strings in 8, 9, 23, of Ex. VII.

4. Find the pressures on the water in 2, 4, 5, of Ex. VIII.

5. Prove that in Attwood's machine, if the total mass of the moving bodies be constant, the greater the tension of the string is, the less is the acceleration.

6. A body of 5 kilograms, moving with a velocity of 3 kilotachs, impinges on a body of 3 kilograms moving with a velocity of 1 kilotach; If $e = \frac{2}{3}$, find the velocities after impact.

7. Two bodies of unequal masses, moving in opposite directions with momenta equal in magnitude, meet; shew that the momenta are equal in magnitude after impact.

8. Two wooden balls whose masses are 12 and 16 ozs. are made to impinge directly on one another. One of the balls is furnished with a spike to prevent the rebo ind. If the velocity of the lighter be 12 ft. per sec., what must that of the larger be that the motion may be destroyed by the impact.

9. The result of an impact between two bodies moving with equal velocities in opposite directions, is that one of them turns back with its original velocity, and the other follows it with half that velocity; find e and the ratio of the masses.

10. A bomb-shell moving with a velocity of 50 ft. per sec. bursts into two parts whose masses are 70 and 40 lbs. After bursting, the larger part turns back with a velocity of 10 ft. per sec.; find the velocity of the smaller part.

11. A and B are two uniform spheres of the same material and of given masses. If A impinges directly upon a

third sphere C at rest, and then C on B at rest, find the mass of C in order that the velocity of B may be the greatest possible for a given initial velocity of A.

12. Find the necessary and sufficient condition that one body moves after direct impact with the original velocity of the other.

13. Two bodies, whose masses are as 10 to 1, start from rest and move under the action of their mutual attraction. At one instant the velocity of the greater mass is 100; find the velocity of the smaller mass at the same instant.

14. A number (n) of balls whose masses form a G. P. whose common ratio is $1 \div e$, hang close to one another with their centres in one line. The first is then made to strike the second with a given velocity, in consequence of which the second strikes the third, the third the fourth, &c.; find the resultant motions of the balls.

15. A strikes B, which is at rest, and after impact rebounds with a velocity equal to that of B; shew that B's mass is at least 3 times A's mass.

16. When two uniform spheres impinge directly, find the changes of visible kinetic energy during each of the stages of impact.

17. If the sum of the masses of the impinging bodies be constant, find when the loss of visible kinetic energy will be a maximum; what will the loss then be?

18. Two cannon balls of masses 20 and 50 lbs. meet one another with velocities of 40 and 20 ft. per sec. respectively; find their velocities and momenta after impact, $(e = \frac{5}{9})$, and the visible kinetic energy lost by the impact.

Chapter XIII.

Dimensional Equations.

106. In the previous pages the student has been introduced to two distinct scientific systems of units, called the C. G. S. and F. P. S. systems respectively. In both systems three independent or fundamental units are chosen, and from these all others are derived. It is not necessary that any three special units must be taken as the fundamental ones. The three, however, which are most easily fixed upon: and with standards of which, comparisons are most easily and directly made, at all times and at all places: and in relation to which the derived units are most easily defined, and are of the simplest *dimensions*, in virtue of the established relations between the different units: are the units of length (or distance), mass, and time.

The relations between the derived and fundamental units, already expressed by algebraical formulae, leads us to the the consideration of *dimensional equations*, or such as express how the derived units depend upon the fundamental units.

Whatever units of length, time, and velocity be used, $V \propto L \div T$, where V measures the velocity of a body moving uniformly, and L is the distance passed over by the body in the time T. Now if we take the unit of velocity as that in which unit of length is passed over in unit of time, the relation is expressed thus, $V = L \div T$. Hence if v, l, t, denote the units of velocity, length, and time in a scientific system, $v = l \div t$. This is called a dimensional equation. It tells us that the unit of velocity involves the unit of length to the first power *directly*, and the unit of time to the first power *inversely*.

Similarly, if m, a, f, w, h, denote respectively the units of mass, acceleration, force, work, and rate of working, in a scientific system,

$$a=\frac{v}{t}=\frac{l}{t^2}\,,\quad f=ma=\frac{ml}{t^2}\,,\quad w=fl=\frac{ml^2}{t^2}\,,\quad h=\frac{w}{t}=\frac{ml^2}{t^3}$$

The last equation tells us that the unit of rate of working involves the unit of mass to the first power, and the unit of length to the second power, *directly*, and the unit of time to the third power *inversely*.

If θ denote the unit of angle, $\theta=\frac{\text{arc}}{\text{radius}}=\frac{l}{l}=l^0$, *i.e.* the unit of angle is independent of the fundamental units.

If s, p, denote the units of surface and pressure-intensity,

$$s=l^2, \text{ and } p=\frac{f}{s}=\frac{m}{lt^2}$$

If b, d, denote the units of bulk and density,

$$b=l^3, \text{ and } d=\frac{m}{b}=\frac{m}{l^3}$$

In whatsoever way the dimensions of a derived unit be deduced, they must of necessity always be the same. Thus it has been proved that the visible kinetic energy of a body whose mass is M and velocity V is $\frac{1}{2}MV^2$, therefore the dimensions of energy or work must be mv^2, *i.e.* $ml^2 \div t^2$, as above proved. Again, the unit of force is that which generates unit of momentum (mv) in unit of time; therefore the dimensions of force are $mv \div t$, *i.e.* $ml \div t^2$, as above proved.

107. An important use of dimensional equations is to facilitate the calculation of the numerical relations between the derived units of different systems, when the numerical

relations between the fundamental units are known. Thus if l', m', t' denote the fundamental units in the F. P. S. system, and p' the derived unit of pressure-intensity,

$$p' = \frac{m'}{l't'^2}, \; p = \frac{m}{lt^2}, \; \therefore \; \frac{p'}{p} = \frac{m'}{m} \cdot \frac{l}{l'} \cdot \left(\frac{t}{t'}\right)^2$$

$= 453{\cdot}593 \times 0{\cdot}0328087 = 14{\cdot}8818$ (see tables art. 108), *i.e.* 1 poundal per square foot $= 14{\cdot}8818$ prems.

Ex. Find the fundamental units in a scientific system in which a mile per hour is the unit of velocity, a pound-weight the unit of force, and a foot-pound the unit of work.

Let L, M, T, denote the fundamental units:

$$\frac{L}{T} = \frac{5280}{3600} \cdot \frac{l'}{t'} = \frac{22}{15} \cdot \frac{l'}{t'} \quad \ldots\ldots \quad \ldots \quad \ldots\ldots \quad (1)$$

$$\frac{ML}{T^2} = 32\tfrac{1}{6} \times \frac{m'l'}{t'^2} = \frac{193}{6} \cdot \frac{m'l'}{t'^2} \quad \ldots\ldots \quad \ldots\ldots \quad \ldots \quad (2)$$

$$\frac{ML^2}{T^2} = \frac{193}{6} \cdot \frac{m'l'^2}{t'^2} \quad \ldots \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad (3)$$

$$\therefore \; L = l' = 1 \text{ foot}, \; T = \frac{15}{22} t' = \frac{15}{22} \text{ second, and}$$

$$M = \frac{193}{6} \left(\frac{15}{22}\right)^2 . \, m' = 14 \frac{923}{968} \text{ pounds.}$$

108. The following tables give the numerical relations between the C. G. S., F. P. S., and a few other frequently occurring units. The numbers in the tables of length and mass give the results of the most accurate observations made in the comparisons of the French and English standards of measurement. Those in the other tables are calculated from the dimensional equations of the units, as explained in last article. Each number is true to the last decimal place given, and the mantissae of the logarithms of the true ratios are added.

I. Length or Distance.

				Mantissae.
1 inch	=	2·53998	centimetres	4048298
1 foot	=	30·4797	"	4840111
1 yard	=	91·4392	"	9611324
1 mile (statute)	=	160933	"	2066451
1 decimetre	=	3·93704	inches	5951702
1 metre	=	3·28087	feet	5159889
1 kilometre	=	0·62138	mile	7933549

II. Area or Surface.

1 square inch	=	6·45148	square centime's	8096597
1 " foot	=	929·014	" "	9680222
1 acre	=	40·4678	ares	6071101
1 square mile	=	2·58994	square kilometres	4132901
1 square decimetre	=	15·5003	square inches	1903403
1 are	=	1076·41	" feet	0319778
1 square kilometre	=	247·110	acres	3928899
1 " "	=	0·38611	square mile	5867099

III. Volume, Bulk, or Capacity.

1 cubic inch	=	16·3866	cubic centimetres	2144895
1 " foot	=	28·3161	litres	4520332
1 gallon	=	4·54102	"	6571531
1 litre	=	61·0254	cubic inches	7855105
1 decalitre	=	0·35316	cubic foot	5479668
1 "	=	2·20215	gallons	3428469

IV. Angle.

			Mantissae.
1 degree, or 1°	=	0·0174532925 radian	2418774
1 right angle	=	1·5707963268 "	1961199
1 radian	=	57·295779513 degrees	7581226
π	=	3·1415926536	4971499
$1 \div \pi$	=	0·3183098862	5028501
π^2	=	9·8696044011	9942997
$1 \div \pi^2$	=	0·1013211836	0057003

V. Mass.

1 grain	=	0·064799 gram	8115679
1 ounce avoir.	=	28·34954 "	4525461
1 pound "	=	453·5927 "	6566661
1 ton	=	1016048 "	0069141
1 gram	=	15·43235 grains	1884321
1 kilogram	=	2·204621 lbs. avoirdupois	3433339
1 tonne	=	0·984206 ton	9930859

VI. Density.

1 lb. avoir. per cub. ft.	=	0·01602 gram per cub. cm.	2046328
1 gram per cub. cm.	=	62·4262 lbs. av. per cub. ft.	7953672

VII. Time.

1 day (mean solar)	=	86400 seconds	9365137
1 sidereal day	=	86164·1 "	9353264
1 mean sidereal month	=	2360591·5 "	3730210
1 " " "	=	27·321661 days	4365071
1 mean synodic "	=	29·530589 "	4702721
1 sidereal year	=	31558149·6 seconds	4991116
" "	=	365·2564 days	5625978
1 mean tropical year	=	365·2422 "	5625809

Note. A *solar day* is the time in which the sun apparently revolves around the earth. A *sidereal day* is the time of the apparent rotation of the sphere of the heavens. A *sidereal month* is the time in which the moon makes a complete revolution in the sphere of the heavens amongst the fixed stars. A *synodic month* is the time between two consecutive full moons. A *sidereal year* is the time in which the sun apparently makes a complete revolution in the sphere of the heavens amongst the fixed stars. A *tropical year* is the time between two consecutive appearances of the sun on the *vernal equinox*, one of the points in which the *equinoctial* cuts the *ecliptic*; it governs the return of the seasons, and varies slowly through a maximum range of about a minute on each side of the mean value. The student will do well to satisfy himself that a real *positive* (+) rotation of the earth would produce an apparent *negative* (−) rotation of the sphere of the heavens, and a *positive* (+) revolution of the earth around the sun would produce an apparent *positive* (+) revolution of the sun around the earth. It follows from this that the number of sidereal days in a sidereal year exceeds the number of mean solar days by unity; whence the relation between these days.

VIII. Velocity.

			Mantissae.
1 foot per second	=	30·4797 tachs	4840111
1 mile per hour	=	44·7036 "	6503425
1 " "	=	22÷15 or 1·4$\dot{6}$ ft. per sec.	1663314
1 hectotach	=	3·28087 ft. per sec.	5159889
1 kilometre per hour	=	250÷9 or 27·$\dot{7}$ tachs	4436975

IX. Momentum.

1 lb. 1 ft. per sec.	=	13825·4 gramtach	1406772
1 gramtach	=	6·07578 grains 1 in. per sec.	7836022

X. Force (taking g = 980·54).

			Mantissae.
1 poundal	=	13825·4 dynes	1406772
1 megadyne	=	72·3307 poundals	8593228
1 kilodyne	=	15·7386 grains-weight	1969668
1 "	=	1·01985 gram-weight	0085347
1 grain-weight	=	63·5380 dynes	8030332
1 pound-weight	=	444766 "	6481314

XI. Pressure-intensity.

1 poundal per sq. ft.	=	14·8818 prems	1726550
1 decaprem	=	0·67196 poundals per sq. ft.	8273450
1 lb.-wt. per sq. in.	=	70·3083 grams-wt. " cm.	8470064
1 ton-wt. per sq. ft.	=	$15{\cdot}555\dot{5}$ lbs.-wt per sq. in.	1918855
1 mean atmosphere	=	1·01360 megaprem	0058672
(= 76 cm. of mercury	=	1033·0 gram-wt. per sq. cm.	0142248
at o° at the latitude	=	14·6967 lbs.-wt. per sq. in.	1672184
of Paris)	=	0·94478 ton-wt. per sq. ft.	9753329

XII. Work and Energy.

1 foot-poundal	=	421394 ergs	6246883
1 million ergs	=	2·37308 foot-poundals	3753117
1 foot-pound	=	0·13825 kilogrammetre	1406772
1 kilogrammetre	=	7·23307 foot-pounds	8593228
1 Watts' horse-power	=	7455·99 megadyntachs	8725052
1 force-de-cheval	=	7354·05 " "	8665266

g	= 980·54 tachs per sec. at Kingston, Ont.	9914653
	= 980·61 tachs per sec. at lat. 45°, and the mean value over the earth's surface.	9914963
	= 980·94 tachs per sec. at Paris.	9916424
	= $32\frac{1}{6}$ ft. per sec. per sec. at Kingston, Ont., and the mean value over the earth's surface	5074061
	= 32·2 ft. per sec. per sec. the mean value in the British Isles.	5078559

Examination XIII.

1. What determines the choice of fundamental units?

2. Why is the French method of forming multiples and submultiples of standard units the best?

3. Define a dimensional equation. Write down the dimensional equations of angular velocity, momentum, energy, angle, pressure-intensity, and density.

4. Determine the ratios of the units of acceleration, angular velocity, density, and the gravitation units of the rates of doing work, in the F. P. S. and C. G. S. systems.

5. Define the following terms: mean solar day, sidereal day, sidereal month, synodic month, sidereal year, tropical year, equinox, equinoctial, ecliptic.

6. How is the ratio of the sidereal day to the mean solar day determined? Calculate the ratio.

7. Check all the ratios given in tables II., III., VI., and VIII. to XII. of art. 108.

Miscellaneous Examples.

1. In a scientific system, the unit of velocity is 1 kilometre per hour, the unit of acceleration is g (981), and the unit of force is the weight of a kilogram; find the fundamental units, and the substance of unit density, in terms of the C. G. S. units.

2. A saw-mill was driven by an engine of 3 H. P., and in 10 minutes 12 square feet of green oak were sawn by the mill; find the modulus of the mill. (See 15, Ex. XI).

3. Name the absolute and gravitation units of weight, according to both the C. G. S. and F. P. S. systems.

4. Two bodies of 9 and 5 grams draw by their weight a body of 7 grams over a smooth pully. After moving for 2 seconds, the 9 grams are removed without disturbing the motion; find how long the 7 grams will continue to rise.

5. If a metre be the unit of length, 10^{-4} of a day the unit of time, and a kilogram the unit of mass; find in dyntachs the derived unit rate of working, and in prems the unit of pressure-intensity.

6. A solid body balances 108 grams in air, 71 in water, and 76 in alcohol; find to an approximation of the second degree the specific gravities of the solid body and alcohol, the specific gravity of the standard masses being 8·4.

7. Oxygen at 0° and 76 cm. pressure has density 1·1056 with respect to air; find its density at 100° and 70 cm., 1) with respect to air at 0° and 76 cm., 2) with respect to air at 100° and 70 cm.

8. A flask of 2 litres capacity was found to balance 1·6 grams more, when filled with carbonic acid, than when filled with air at 0°; find the presure of the atmosphere, given the specific gravity of air and of carbonic acid (p. 71).

9. Define *equal times*, and state in terms of your definition the great physical truth contained in Newton's First Law of Motion.

10. Given the densities of air and oxygen (p. 71), find at what pressure the density of oxygen at – 30° C will be 1·5 with respect to air.

11. A cube of metal has an edge of 1 metre, and density 10; find its volume, mass, and weight at Washington.

12. Find in dynes the apparent weight of a body of 240 grams mass and of volume 1 litre, when weighed in air at – 30° C and 80 cm. pressure at Kingston, Ont.

13. When a body is moving in a curved path with a variable velocity and variable acceleration, what is meant by its direction of motion, velocity, and acceleration at any instant?

14. The weight of a body at Paris is 98094 dynes, what would be the mass of that body at the surface of the sun? Would the weight there be the same as at Paris?

15. A sunken vessel, whose bulk is a megalitre and mass 10^6 kilograms, is to be raised by attaching water-tight barrels to it. If the mass of each barrel be 30 kilograms, and the volume a kilolitre, find how many will be required.

16. If a mercurial barometer of 1 sq. in. section stand at 30 inches, what will be the height of a sulphuric acid barometer of section $1 \div 1{\cdot}84$ sq. in.? Given the densities of mercury and sulphuric acid (p. 71).

17. Find the height of the water barometer under the mean atmospheric pressure, when the temperature is 15° C; given the tables at pp. 71 and 76.

18. If the temperature be 15° at Edinburgh, and the coefficients of dilatation of the barometer scale and mercury be $1{\cdot}8 \times 10^{-5}$ and $1{\cdot}794 \times 10^{-4}$; find the barometric reading when the pressure of the atmosphere is a megaprem.

19. What effect has the dilatation of the glass tube on the height of the barometric column? Explain your answer.

20. If a force-de-cheval be the unit rate of working, g the unit of acceleration, and the weight of a kilogram the unit of force; find the units of pressure-intensity and momentum.

21. A shaft a metres deep is full of water; find the depth of the surface of the water when $\frac{1}{2}$ of the work required to empty the shaft has been done.

22. The mass of a specific gravity bottle is 20·5 when empty, 70·5 when filled with water, 63 when filled with turpentine; when 10 grams of salt are put into it, and it is thereafter filled up with turpentine, the mass is 69·6; find the s. g. of the turpentine, and of the salt to an approximation of the first degree.

23. A number n of balls A, B, C,...... formed of the same substance, are placed in a straight line. A, whose mass is m, is then projected with a given velocity u so as to impinge on B; then B impinges on C; and so on; find the masses of B, C,...... so that each ball may be at rest after impinging on the next, and find also the velocity of the last ball.

24. A uniform force acting on a body for one-tenth of a second produces a velocity of a mile per minute; compare the force with the weight of the body at Greenwich.

25. One end of a string is fastened to a body of 10 kilograms; the string passes over a fixed pully, then under a movable pully, and has its other end attached to a fixed hook; $7\frac{3}{4}$ kilograms are attached to the movable pully whose mass is 250 grams; if the three parts of the string be parallel, and friction and the masses of the string and fixed pully may be neglected, find the accelerations of the masses and the tension ot the string.

26. A ball is let fall from a height of 100 metres, and strikes a horizontal surface ($e = \frac{2}{3}$); find how high the ball will rise again, neglecting the resistance of the air.

27. A body of 100 kilograms pulls by its weight 200 kilograms along a rough horizontal plane; if the coefficient of friction be 0·2, find the velocity after moving through a distance of a hectometre.

28. Two bodies of different volumes have the same weight in water. Will their weights be the same in air and in mercury? If not, how will they differ?

29. A stream is a feet broad, b feet deep, and flows at the rate of c feet per hour; there is a fall of d feet; the water turns a machine of which the efficiency is e; it requires f foot-pounds per minute for 1 hour to grind a bushel of corn; determine how much corn the machine will grind in 1 hour.

30. Find what will the volume of a litre of air at 0° and under the mean atmospheric pressure become, when at the bottom of the deepest known part of the ocean (s. g. 1·027) viz., 5 miles, and what will be its mass at the same place?

31. Find the visible energy of the boulder in 1, Ex. XII. What becomes of it when the boulder strikes the ground?

32. Find what must be the area of a cake of ice (s. g. 0·91674), 18 inches thick, sufficient to bear the aggregate weight of three school boys whose aggregate mass is 280 lbs.; 1) in fresh water, 2) in sea-water, (see tables, art. 81 and 108).

33. Three bodies P, Q, R, of masses 30, 15, 10 kilograms respectively, are connected by strings AB and BC, whose lengths are 5 m. and 70 cm. Q, R, BC, and half of AB lie on the edge of a table vertically under a peg, over which the other half of AB is placed holding P. If P be now allowed to fall freely, find the motions of P, Q, and R, the tensions of the strings after both become stretched, and the measures of the impulsive tensions which set Q and R in motion. Friction and the masses of the strings may be neglected, and $g = 980$.

Answers to the Exercises.

When no unit is appended to an answer, the units of the C. G. S. or F. P. S. system are to be understood. When the correct answer cannot be expressed exactly in the usual notation, or only by a large number of decimal places, the answer given is true to the last figure.

Exercise I.

1. 485·85 ares; 904·31 ares; $2·80195 \times 10^9$ litres.
2. 12741 kilom.; $5·1002 \times 10^{12}$ ares; $1·0831 \times 10^{12}$ cub. kilom. 3. $4·075 \times 10^9$ cub. kilom.; 1 : 266.
4. $1·60018 \times 10^9$ ares. 5. 1 : 30. 6. If the earth's surface be unity, 0·3987491 : 0·518311 : 0·0829399; torrid zone $2·0338 \times 10^8$ sq. kilom.; temperate zones $2·6436 \times 10^8$ sq. kilom.; frigid zones $4·2303 \times 10^7$ sq. kilom. 7. 28845; 74° 19′ 14″·6.
8. 111·8; 25416; 261·8 litres. 9. 4 : 1. 10. 168·6.
11. 56° 26′ 34″. 12. 252062·5. 13. 1 ÷ 20, or 2° 51′ 53″·2.
15. 31·464 sq. m. 16. 23307; 60·2585. 17. 3631·1; $5·43205 \times 10^8$. 18. 28689·5; $6·54991 \times 10^7$.
19. 3° 49′ 13″·5; 10·002. 20. 1·299038; 2; 2·377641; 2·598076; 2·828427.

Exercise II.

1. 100 min. 2. 72 tachs; 65 tachs. 3. 12000.
4. [illegible] ft. 5. 6607·75 m. 6. 11952 m. 7. 500; $333·\dot{3}$.
8. 33·8. 9. $10\frac{2}{7}$ sec. 10. $2\frac{4}{7}$ cm. 11. $\pi \div 45$.
12. $\pi \div 43200$. 13. $22·\dot{2}$. 14. 1 m.; 2π sec.

Exercise III.

1. 432000. 2. 127072800. 3. 139104000. 4. 2078 upwards; 1844 downwards. 5. $\frac{11}{20}$. 6. 144 : 1.
7. 1 : 400. 8. 100. 9. $\frac{1}{10}$ sec. 10. 10 m.; 10 min.
11. No difference. 12. 30. 13. 1 m. 14. $1{\cdot}0\dot{6}$.
15. 10·8. 16. 700. 17. 18 min. 18. 5 in. 19. $m^3 : n^3$.
21. 15·3.

Exercise IV.

1. 396·9 m. 2. 4·15 sec. 3. 10 sec.; 1600 ft.; $3\frac{1}{8}$ sec., or $16\frac{7}{8}$ sec., 843 ft. 9 in. high; 3·9 sec.; 195·96; 112 ft. 4. 112·25. 5. 54·8 sec. 6. 6·9 sec.
7. 793. 8. 10. 9. 980. 10. 50. 11. 4 h. 18 min. 59·8 sec.; 47849·6 m. 12. 490·25 m.; 9904·5.
13. 1000, −100. 14. $b^2 - a^2 : d^2 - e^2$. 15. 10^5.
16. 164·15. 17. 78·44 m. 18. 1 sec. 19. $\pi : 4$.
20. 10 sec.; 1·9 m. 21. 11 ft. 23. 2 sec. (take $g = 980$).
24. 2800; 2800, 0. 26. $b - a$.

Exercise V.

1. 8100. 2. 28498552 kilogrs. (see p. 107). 3. 4·6 kilogrs.; 2 litres. 4. 82·7 grs. 5. 2·66.
6. $6{\cdot}1418 \times 10^{21}$ tonnes. 7. 8·02. 8. 140·55 kilogrs.; 1·634. 9. 260·42. 10. 1·02605. 11. 13·6.
12. 4 : 3. 13. $13{\cdot}\dot{3}$ lbs. 14. 8 : 9.

Exercise VI.

1. 200; 10 m. 2. 650. 3. 0·191. 4. 500.
5. $6{\cdot}6584 \times 10^{21} : 1$. 6. 135·9 : 1. 7. $0{\cdot}2\dot{7}$.
8. 1 : 105·894. 9. 6000. 10. 1 : 3231. 11. 77·76 kilogrs. 12. 10 kilodynes; 1 kilotach. 13. 2 : 1; 1 : 2. 14. 4 : 1. 15. 60 : 11. 17. 1 : 1; 400 : 19.

Exercise VII.

1. 9805; 98050. 2. 12·5 kilogrs. 3. 4·2721 kilogrs.
4. 14707500; 7353·75. 5. 4902500; 392·2. 6. 31313 tachs. 7. 7·772 lbs.-wt. 8. 10 sec.; 183·35 ft.
9. 6 lbs. or $16\frac{2}{3}$ lbs. 10. 650; 10·2 milligrams-wt.
11. 0·032 sec. 12. 980·5 cm. 13. 980·5 grs.
14. 143·793; 385·37; 377852. 15. 100·795 megaprems + pressure of the atmosphere. 16. 707·$\dot{6}$; 321·$\dot{6}$; 128·$\dot{6}$. 17. 797·2. 18. 900 milligrs.; 899·08.
20. 898·01 grams-wt. 21. 521·$\dot{6}$. 22 200 ft. 23. 10 m.

Exercise VIII.

1. 2·735. 2. 0·515. 3. 8·12; 0·164. 4. $\frac{1}{4}$. 5. 20·0$\dot{6}$ lbs.-wt. 6. 388 grs. 7. 61 ÷ 136. 8. 0·895.
9. 2153 grs.-wt. 10. 1661·808 grs.-wt. 11. 894112 dynes. 12. 74·31$\dot{3}$ kilogrs.-wt. 13. 15 ÷ 31.
14. 13·0206 kilogrs.-wt. 15. 87 cub. cm. 16. $\frac{1}{2}$; $\frac{5}{4}$.
17. 22·$\dot{6}$; 9·0$\dot{6}$; 8·9 lbs. 18. 5·848 cub. ft. 19. 2 : 3; 5 : 4; 2. 20. 1778 grs.-wt. 21. 3421·9 grs. 22. 1·03.

Exercise IX.

1. 13·9218 grs.; 13650·8 dynes. 2. $1{\cdot}24112 \times 10^6$.

Exercise X.

1. 252062·5 litres; 312143; $3{\cdot}06069 \times 10^8$. 2. 0·25; 0·250969; 0·250755. 3. 19·2931, 0·720012; 19·2695, 0·720374; 19·2663, 0·720367.

Exercise XI.

1. $5{\cdot}616 \times 10^7$ ft.-lbs. 2. $0{\cdot}\dot{2} \times g \times 10^6$.
3. $3{\cdot}92216 \times 10^{14}$. 4. 120373. 5. 32 tonnes. 6. 4.$\dot{6}$.

7. 7212535 ft.-lbs.; $1{\cdot}12348 \times 10^7$ ft.-lbs. 8. 14·2, (1 gal. of water = 10 lbs). 9. 3916·8. 10. For 3920 bricks, 3487·65 ft.-lbs. per min. 11. $9{\cdot}5\dot{4}$ cts.
12. $32\frac{13}{16}$ miles per hour; 8064 ft. 13. 4 min. 39·3 sec.; 1 mile 480 yds. 14. 1 hr. 37 min. 23·3 sec.
15. 2·49233. 16. 0·756. 17. 3·177 effective H. P.; 24192.
18. $9{\cdot}87 \times 10^{10}$. 19. $1{\cdot}5114 \times 10^7$. 20. $3{\cdot}75 \times 10^9$.
21. 55·22 cub. ft. per min. 22. 8 tons 2207·7 lbs.
23. $1{\cdot}125 \times 10^{10}$. 24. 12 ÷ 4825. 25. $20\frac{4}{11}$ hrs.
26. $1{\cdot}66795 \times 10^{11}$ ft.-lbs.

Exercise XII.

1. $1{\cdot}37575 \times 10^{-10}$ cm. per year; $2{\cdot}9776 \times 10^{-17}$ cm.
2. 6·6 ft. per min. 3. 2·4 lbs.-wt.; 7·5 lbs.-wt., or 12·5 lbs.-wt.; tension of the string connecting the 4 and 6 kilogrs., at first 7·2 kilogrs.-wt., afterwards, 4·8 kilogrs.-wt.; tension of string connecting the 4 and 5 kilogrs., 4 kilogrs.-wt. 4. 48·72; 6 lbs.-wt.; $119{\cdot}9\dot{3}$ lbs.-wt. 6. 1750; $3083{\cdot}\dot{3}$. 8. -9. 9. $\frac{1}{4}$; 1 : 4. 10. 155. 11. $C = \sqrt{(AB)}$.
12. Ratio of masses $e : 1$. 13. 1000. 14. They are all left at rest except the last, which moves away with a velocity e^{n-1} times that of the original velocity of the first ball. 16. $m_1 m_2 (u_1 - u_2)^2 \div 2\,(m_1 + m_2)$; $e^2 m_1 m_2 (u_1 - u_2)^2 \div 2(m_1 + m_2)$. 17. When $m_1 = m_2$; $\frac{1}{4} m_1 (1 - e^2)(u_1 - u_2)^2$. 18. $-26{\cdot}\dot{6}, 6{\cdot}\dot{6}; -533.\dot{3}, 333{\cdot}\dot{3}$; 552·677 ft.-lbs.

Miscellaneous Examples.

1. $10^6 \div (36^2 \times 981)$ cm.; 1 kilogr.; $10^3 \div (36 \times 981)$ sec.; substance whose density relatively to water is $36^6 \times 981^3 \div 10^{15}$. 2. 0·353. 4. 4 sec.

5. $10^{13} \div 864^3$; $10^5 \div 864^2$. 6. 2·91644; 0·86504.
7. 0·982333; 1·1056. 8. 90·4 cm. 10. 91·778 cm.
11. 10^6; 10^7; $9{\cdot}8008 \times 10^9$. 12. 233830.
14. 100 grs.; no. 15. (bulk, half a megalitre), 516.
16. 18 ft. 5·674 in. 17. 10 m. 16 cm. 18. 75·12.
20. $g^3 \div 10^6 \times 75^4$ prems; 7·5 megagramtach.
21. $a \div 2$. 22. 0·85; 2·36. 23. $A \div e$, $A \div e^2$, ... ; $e^{n-1}u$. 24. 1 : 131·69. 25. $g \div 2$; $g \div 4$; 5 kilogrs.-wt. 26. 44·4m. 27. 1980·44 tachs.
29. $62{\cdot}4\, abcde \div 60f$. 30. 1·25 cub. cm.; 1·2932.
31. $1{\cdot}79 \times 10^{14}$ ergs. 32. 35·914 sq. ft.; 27·12 sq. ft.
33. Q starts with vel. $1400 \div 3$, R with 420; $27{\cdot}\dot{2}\dot{7}$ and $10{\cdot}\dot{9}\dot{0}$ kilogrs.-wt.; impulsive tension of AB when Q was set in motion, 7 megagramtachs; when R was set in motion, impulsive tension of AB was 2·8 megagramtachs, and of BC 4·2 megagramtachs.

CORRIGENDA.

p. 7, Ex. 14, *after* $\frac{\pi}{6}$, *insert* and one of the angles $\frac{2\pi}{3}$,

p. 26, 3)., *instead of the second line, read* $= \frac{(3922)^2}{2 \times 980{\cdot}5} = 7844$

p. 47, Ex. 15, *for* 11 to 15, *read* 22 to 15.

p. 110, line 14, *for* 1033·0 *read* 1033·30.

p. 113, Ex. 15, *after* bulk is, *insert* half.

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